MATHEMATICS WITH CALCULUS

Year 13

MATHEMATICS DEPARTMENT
Central School - Port Vila (2020)

Table of Contents

INSTRUCTIONS

This package contains:

1- **A scheme of work (or Overview)**
2- **Lesson activities**
3- **Solutions at the end each lesson activity**

Please read the instructions carefully before you start.

Follow this order to work on the daily lesson activities.

Step 1: Read the scheme of work carefully and identify daily activities

Step 2: Read and understand the lesson activity and take note of the specific learning outcomes of each lesson.

Step 3: Redo all the examples provided for better understanding.

Step 4: Do the assigned activities in your exercise book. Show the working out.

Step 5: Check your answer with the solutions provided (*Work on the question before checking your answer with the provided solutions*).

Repeat steps 1 to 5 for each lesson activity.

Please note:

- Spend at least 1 hour to 1 hour and a half on the lesson activity each day.
- Write answers in your books and NOT on the handouts
- Show your all your working outs
- Complete daily activities before moving to the following day's load.
- Be up to date with your work.
- Be responsible for your own work and manage your time wisely!
- Be an independent learner.

Also note that the skill levels are in red.

L1 – skill level 1
L2 – skill level 2
L3 – skill level 3
L4 – skill level 4

Contact your teacher if you have any questions or need clarifications.

Miss. Carla LISHI (Yr13A) Contact: 7756840 or 5990100 Email: clishi23@gmail.com	Mr. Randall TAFORUA (Yr13B) Contact: 5397600 Email : rtaforua@gmail.com

MATHEMATICS WITH CALCULUS - SCHEME OF WORK

Week/ Term	Days	Strand and sub strand	Specific learning outcomes (SLO)	Activities
W9 – T1 (13/03 – 3/04)	Monday	1.0 ALGEBRA 1.1 Basic algebra skills	Cal1.1.2.3	**Lesson activity 7 (p.36) PARTIAL FRACTIONS** - Read notes and study examples (p.36-41) - Redo examples - Do activity 7A. a-c (p.41)
	Tuesday		Cal1.1.3.1	- Study type 2 (p.42) - Redo example - Do 7B (p.43) a, d
	Wednesday			- Study type 3(p.43) - Redo example E - Do activity 7C. a, b (p.44)
	Thursday			- Study type 4 (p.45) - Redo example1 - Do 7D.a,b,c (p.46)
	Friday		Cal1.1.2.7 Cal1.1.3.3	**Lesson activity 8 MATHEMATICAL INDUCTION** - Read and study p.47-48 - Redo example A for better understanding - Do 8A. a, b
W10 – Term 1 (6/04-10/04)	Monday	1.0 ALGEBRA	Cal1.1.1.1 3 Cal1.1.2.4 Cal1.1.3.2	**Lesson activity 9: BINOMIAL THEOREM** - Read and study p.49-50 - Redo examples A and B for better understanding - Do 9A. 1.a, 2 and 3c (p.51)
	Tuesday	1.0 ALGEBRA		Work on your IA (You will be informed later.) - *Check your prescription for information on the Task 1.*
	Wednesday	1.0 ALGEBRA		
	Thursday	1.0 ALGEBRA		
	Friday	Good Friday (Public Holiday)		
W11 – Term 1 (13/04 – 17/04)	Monday	Easter Monday (Public Holiday)		
	Tuesday	1.0 ALGEBRA	Cal1.1.1.2 Cal1.2.3.3	**Lesson activity 3 (p.15)** - Ready and study p.15-19 - Study examples - Do activity 3A. 1.a, b; 2

	Wednesday	1.0 ALGEBRA	Cal1.2.3.1	- Do Q3. a, b (p.20)
	Thursday	1.0 ALGEBRA	Cal1.1.1.5 Cal1.1.1.6 Cal1.1.1.7 Cal1.1.1.14	**Lesson activity 4: FACTORISATION** - Read and study p.21-25 - Redo examples given for better understanding
	Friday	1.0 ALGEBRA	Cal1.1.2.5 Cal1.2.4.1	- Do activity 4A, Q1.a, b, Q2.a, c (p.25)
W12 – Term 1 (20/04 - 24/04)	Monday	1.0 ALGEBRA	Cal 1.2.2.6 Cal 1.2.4.2	- Do Q3. a,b (p.25) - Do Q4. a, b (p.26)
	Tuesday	1.0 ALGEBRA	Cal1.1.1.1 1 Cal1.1.1.1 2 Cal1.1.2.2	**Lesson activity 5: REMAINDER AND FATOR THEOREM** - Read and study p.27-28 - Redo examples for better understanding - Do 5A. 1.a, and 2 (p.28)
	Wednesday	1.0 ALGEBRA	Cal1.2.2.5 Cal1.2.3.7	- Read “the remainder theorem” - Redo example - Do activity 5B. Q1, Q4 (p.29)
	Thursday	1.0 ALGEBRA	Cal1.2.4.3	- Read and study “the factor theorem” (p.30) - Redo example A (p.30) and example B (p.31) - Do activity 5C. Q1a,b ; Q2 ; Q3 (p.32) -
	Friday	1.0 ALGEBRA	Cal1.1.1.7 Cal1.1.1.8 Cal1.1.1.9 Cal1.1.1.10 Cal1.2.2.3 Cal1.2.3.4 Cal1.2.3.2 Cal1.2.4.2 Cal1.2.1.3 Cal1.2.1.4 Cal1.2.2.2	**Lesson activity 6: INDICES AND LOGARITHMS** - Read and study p33- 36 - Redo examples A, B, C to better understand the concepts
W13 – Term 1 (27/04 – 1/05)	Monday	1.0 ALGEBRA		- Do activity 7A, Q1.a, c. d (p.36)
	Tuesday	1.0 ALGEBRA		- Do Q2.a,b, g, h (p.36-37)
	Wednesday	1.0 ALGEBRA	Cal1.1.1.18 Cal1.1.2.6	**Lesson activity 10 SURDS** - Read and study example A, B p.52 - Do activity 10A, Q1. a-d

			Cal1.1.1.16 Cal1.1.2.9 Cal1.1.2.4	- Read and study “sums and differences of surds” - Do activity 10B, Q1.a, b (p.53) - Ready and study “multiplying surds” Do activity 10C, Q1.a,b and Q2.a,b
	Thursday	1.0 ALGEBRA		- Read and study “Rationalising surds” and “Conjugate surds” (p.54- 55) - Do activity 10D, Q1.a,b; Q2, Q4.a,c,e
	Friday	Labour day (Public Holiday)		
W1 – Term 2 (18/05 – 22/05)	Monday	1.0 ALGEBRA	Cal1.2.2.4 Cal1.2.3.6	- Read and study “Solving equations involving surds” (p.56) - Do activity 10E, Q1.a,b; Q2.a,b
	Tuesday	1.0ALGEBRA **1.3 COMPLEX NUMBERS**	Cal1.3.1.1 Cal1.3.2.1	**Lesson activity 1 COMPLEX NUMBERS** - Read and study p. 57-59 - Redo examples for better understanding - Do activity 1A, Q1.a, b (p. 59) - Do Q2.a,b (p.59)
	Wednesday	1.0ALGEBRA **1.3 COMPLEX NUMBERS**	Cal1.3.2.2 Cal1.3.3.1 Cal1.3.4.1	Lesson activity 2: CPMPLEX NUMBERS, ARGAND DIAGRAM - Read and study p.61- 62 - Redo examples for better understanding - Do activity 2A- Q1.a,b (p.63) - Do Q2.a
	Thursday	1.0ALGEBRA **1.3 COMPLEX NUMBERS**	Cal1.3.1.2 Cal1.3.1.3 Cal1.3.2.2	**Lesson activity 3 (p.64)** - read and study p. 64-66) - redo examples for better understand - do activity 3A, Q1.a,b (p.67) - Do Q2.a (p.67)
	Friday	1.0ALGEBRA **1.3 COMPLEX NUMBERS**		- Read and study **“MULTIPLICATION AND DIVISION OF A COMPLEX NUMBER IN POLAR FORM.” (p. 67)**

<table>
<tr><td></td><td></td><td></td><td rowspan="3"></td><td>- Redo examples for better understanding</td></tr>
<tr><td rowspan="5">Week 2 (25/05 – 29/05)</td><td>Monday</td><td>1.0ALGEBRA
1.3 COMPLEX NUMBERS</td><td>- Read and study “De Moivre’s theorem” (p.68-70)
- Do activity 3B, Q1.a, b (p.70)</td></tr>
<tr><td>Tuesday</td><td>1.0ALGEBRA
1.3 COMPLEX NUMBERS</td><td>- Do Q2.a,b (p.70)
- Do Q3.a,b (p.70)</td></tr>
<tr><td>Wednesday</td><td rowspan="4">1.3 COMPLEX NUMBERS</td><td rowspan="3">Cal1.3.2.4
Cal1.3.3.2
Cal1.3.3.3</td><td>Lesson activity 4: Roots of complex numbers
- read and study p.72- 75
- redo the examples for better understanding
- do activity 4A. 1.a, b(p.75)</td></tr>
<tr><td>Thursday</td><td>- Do Q2.a,b (p.76)
- Do Q5 (p.76)</td></tr>
<tr><td>Friday</td><td>- Read and study “the conjugate root theorem” (p.76)
- Redo examples
- Do activity 4A. Q1; Q3.a;Q 4.a (p.75)</td></tr>
<tr><td>Week 3 (1/06 – 5/ 06)</td><td></td><td>Cal1.3.4.2</td><td>Lesson activity 5: DE MOIVRE’S THEOREME AND COMPLEX ROOTS
- Read and study p.79- 82
- Redo examples
- Do activity 5A, Q1, Q2</td></tr>
</table>

STRAND 1.0   ALGEBRA

SUB-STRAND 1.1 & 1.2: ALGEBRA BASIC SKILLS, POLYNOMIALS AND NON-LINEAR EQUATIONS

LESSON ACTIVITY 1

SUB-STRAND	SLO NO.	SPECIFIC LEARNING OUTCOMES	SKILL LEVEL	SLO CODE	ACHIEVED
1.1	1	simplify linear equations eliminating fractional terms	1	Cal1.1.1.1	
1.2	7	solve linear equations	1	Cal1.2.1.2	
1.2	14	solve rational equations	3	Cal 1.2.3.6	
1.2	2	form equations based on contextual problems.	2	Cal1.2.2.1	
1.1	3	Solve linear inequations	1	Cal1.1.1.3	

LINEAR EQUATION.

A ***linear equation*** in x is one that can be expressed in the form $\boldsymbol{ax + b = c}$**,** where a, b and c can represent any real number, $a \neq 0$.

When an equation has **fractional expressions** in it, e.g: $\frac{x}{2} + \frac{1}{3} = 5$, the usual first step is to find a common denominator and multiply each term by this number to remove the fraction.

Example 1: Solve the equation $8 - 3x = 7$

Answer:
$$8 - 3x = 7$$
$$-3x = 7 - 8 \qquad \textit{[Subtract 8 from both sides]}$$
$$-3x = -1 \qquad \textit{[Divide both sides by -3]}$$
$$\boldsymbol{x = \frac{1}{3}}$$

Example 2: Solve the equation $\frac{2(x+3)}{5} - \frac{x+7}{3} = \frac{3x-8}{2}$

Answer: The common denominator (**CD)** is $5 \times 3 \times 2 = 30$

$$\frac{2(x+3)}{5} - \frac{x+7}{3} = \frac{3x-8}{2}$$

$$30 \times \frac{2(x+3)}{5} - 30 \times \frac{x+7}{3} = 30 \times \frac{3x-8}{2} \qquad \textit{[Multiplying by CD, 30)}$$

$$12(x+3) - 10(x+7) = 15(3x-8) \qquad \textit{[Cancelling denominators]}$$

$$12x + 36 - 10x - 70 = 45x - 120 \quad \textit{[Expanding]}$$

$$2x - 34 = 45x - 120 \quad \textit{[Simplifying]}$$

$$2x - 45x = -120 + 34$$

$$-43x = -86 \rightarrow x = 2 \quad \textit{[Divide by -43]}$$

ACTIVITY 1A: Linear Equations

Solve each of these equations:

1. $6x = 3x + 12$ L1
2. $5(x+7) = 3(x-2)$ L1
3. $\frac{5}{2}x - 3 = -1$ L1
4. $\frac{x}{3} + \frac{x}{4} = 2$ L1
5. $\frac{3x-1}{2x+5} = \frac{1}{2}$ L1
6. $\frac{y+1}{3} + \frac{y-3}{2} = \frac{y+6}{6}$ L1
7. $\frac{3}{4}(y+2) = \frac{2}{3}(y-3)$ L1
8. $\frac{3}{5}(x+1) + \frac{2}{3}(x-1) = 2$ L1
9. $\frac{7-x}{x+1} - \frac{3x-1}{x-1} = -3$ L1
10. $\frac{x+5}{3-x} + \frac{2}{3} = \frac{-3}{x}$ L1
11. Find the values of x in the following rational equations:

(a) $\frac{2}{x} + \frac{3}{5x} = 1$ L2

(b) $\frac{3}{x} + \frac{2}{x} - \frac{1}{4x} = 1$ L3

(c) $\frac{3}{x+1} - \frac{2}{x} = 0$ L2

(d) Find x if $\frac{1}{x(x+1)} = \frac{k}{x} + \frac{2}{(x+)}$ L3

WORD PROBLEMS

Problem given in words **(word problem)** need to be changed to mathematical equations to be solved. This process is called **mathematical modelling**.

Example 1: Shanes is taller than Jason by 2.4 cm. Jason is taller than Ian by 1.3 cm. the three heights total 452 cm. what are their heights? Solve the problem by letting Ian's height be h

Answer: Let Ian's height be $\boldsymbol{h}$

Jason's height is $\boldsymbol{h + 1.3,}$

Shane's height is $\boldsymbol{h + 1.3 + 2.4 = h + 3.7}$

Since their combined height is 452 cm

$$h + h + 1.3 + h + 3.7 = 452$$

$$3h + 5 = 452$$

$$h = 149$$

Thus **Ian** is **149 cm** tall; **Jason's** height is $149 + 1.3 = \mathbf{150.3\ cm}$ and **Shane's** height is $149 + 1.3 + 2.4 = \mathbf{152.7 cm.}$

ACTIVITY 1B: Solving word problems.

1. For each of the following, **form a mathematical equation** and **solve** the problem using appropriate method. All L2

   a) Notebooks cost $1 more than pens. Sandra purchases five pens and six notebooks. Her total expenditure is $32.40. How much does each pen cost?

   b) One girl has four times as much money as her friend. She gives her friend $12.00 and as a result they now have the same amount of money. How much did each have originally?

   c) My son is 31 years younger than I am. In one year's time my son's age will be one quarter of my current age. How old am I?

   d) The perimeter of a room is 50 cm. The length exceeds the width by 3.5 m. Find the length of the room.

   e) Tina and Jane purchase raffle tickets and agree to share the prize money in the ratio of how much each paid. They win $117.00 and Jane who put in $2.00, gets $26.00. What was Tina's share of the cost of raffle tickets?

   f) Shanes is taller than Jason by 2.4 cm. Jason is taller than Ian by 1.3 cm. the three heights total 452 cm. what are their heights? Solve the problem by letting

      i) Jason's height be h
      ii) Shane's height be h

INEQUATIONS

Inequations are mathematical sentences which use **inequality symbols** $(<, \leq, >, \geq)$ instead of equal signs as in an equation.

A linear inequation in x is one that can be expressed in the form $\boldsymbol{ax + b \leq c}$**,** where $a, b\ and\ c$ can represent any real number and the symbol $(\leq)$ can be replaced by other inequalities.

Solving a linear inequation means finding the values which make the linear inequation true. The technique is similar to that of solving an equation except that the ***inequality sign*** is ***reversed*** when ***multiplying*** or ***dividing*** by a ***negative*** number.
Example A: Solve $5 - 3x \leq 8 - x$, where x is a real number.

Answer $$5 - 3x \leq 8 - x$$

$$-3x + x \leq 8 - 5 \quad \textit{[Simplifying]}$$

$$-2x \leq 3$$

$$x \geq \frac{3}{-2} \quad \textit{[divide by -2 and reversing inequality]}$$

$$x \geq -1.5$$

ACTIVITY 1C: Solving Inequations.

Solve these inequations. Check your answers with other students and with your teacher.

1. $4x > 5 + x$ L1

2. $1 - 4x < 7$ L1

2. $3(x - 1) \geq 2 + 5x$ L1

4. $\frac{3}{2}x + 2 \geq \frac{1}{5}$ L1

5. $\frac{5-3x}{2} \geq -2$ L1

6. $\frac{2x+3}{3} < \frac{3x-1}{2}$ L1

7. $\frac{4}{3}(x - 2) - \frac{1}{2}(x + 3) \geq x + 3$ L1

8. $\frac{4x-1}{3} \geq \frac{x+1}{7} - 1$ L1

9. $\frac{2(1-4x)}{5} + 6 \leq \frac{1-x}{2}$ L1

10. $\frac{x}{5} - \frac{2}{3}(1 - x) \geq \frac{x+1}{2}$ L1

Ans:

LESSON ACTIVITY 2:

SUB-STRAND	SLO NO.	SPECIFIC LEARNING OUTCOMES	SKILL LEVEL	SLO CODE	ACHIEVED
1.2	1	**Determine** the variables in a contextual problem.	1	Cal1.2.1.1	
1.1	4	**Rearrange** a formula to obtain the correct subject.	1	Cal1.1.1.4	

MATHEMATICAL MODELLING

- **Mathematical modelling** involves expressing a relationship as a formula in which mathematical symbols and letters are used to represent variables.

Example A: Three quantities used in accounting are assets (A), proprietorship (P), and Liabilities (L). The relationship between these quantities is that assets are equal to the sum of proprietorship and liabilities. In mathematics form, the relationship can be expressed**:** $\boldsymbol{A = P + L}$

CHANGING THE SUBJECT OF A FORMULA.

The **subject** of a formula is a variable which appears alone on one side of the equal sign. The subject is expressed in terms of the other variables. Changing the subject of a formula means rearranging the formula (using inverse operations) so that another variable becomes the subject.

Example B: Make x the subject of $y = 2x - 6$.

To make x the subject of the formula, the given formula $y = 2x - 6$ is to be rearranged using inverse operations such that x becomes the subject, i.e. x is on its own on one side of the equation.

Answer:

$$y = 2x - 6$$

$$y + 6 = 2x$$

$$2x = y + 6$$

$$x = \frac{y+6}{2}$$

NB: Sometimes the variable to be made subject appears more than once in the formula. You will need to collect terms in this variable on one side and then factorise.

Example C: Make T the subject of $\frac{AT+D}{T} = E$

Answer:

$$\frac{AT+D}{T} = E$$

$$AT + D = ET \quad \text{[Multiplying both sides by T]}$$

$$ET = AT + D \quad \text{[Swapping sides]}$$

$$ET - AT = D \quad \text{[collecting terms in T on one side]}$$

$$T(E - A) = D \quad \text{[factorising]}$$

$$T = \frac{D}{E-A}$$

USING A FORMULAE

Formulae need to be used accurately. Using formulae often means **substituting** numbers for some of the variables and solving the equation to **find the value of an unknown variable.**

Example D: The volume of a cone is one third of the product of its height and the area of its base.

i) Express this relationship in mathematical form
ii) Find the volume when $R = 2.152cm$ and $H = 4.365\ cm$
iii) Find the radius of the base when the volume is $63.25\ cm^2$ and the height is 5.230 cm.

Answer:

i) Let the volume of the cone be v, the radius of the base be R, and the height be H.
The base area is πR^2 *[formula for the area of a circle]*
Volume is $\boldsymbol{v = \frac{1}{3}\pi R^2 H}$

ii) $\boldsymbol{v = \frac{1}{3}\pi R^2 H}$ *[In this example, $\boldsymbol{\pi}$ is taken to be 3.142]*

$$v = \frac{1}{3} \times 3.142 \times (2.152)^2 \times 4.365$$

$$\boldsymbol{v = 21.17cm^3}$$

iii) $\boldsymbol{v = \frac{1}{3}\pi R^2 H}$

$$63.25 = \frac{1}{3} \times 3.142 \times R^2 \times 5.230$$

$$R^2 = \frac{3 \times 63.25}{3.142 \times 5.230}$$

$$R = \sqrt{\frac{3 \times 63.25}{3.142 \times 5.230}}$$

R = 3.40 cm (2dp)

<u>ACTIVITY 2A</u>: Writing Formulae, Changing the subject of the formulae & Using Formulae.

1. A rectangle has a length L metres and width W centimetres. Write an expression for: L1
   a) The area in cm^2
   b) The area in m^2
   c) The perimeter in mm

2. Make M the subject of the following formulae: L1

   a) $\frac{2M}{P} = 5$
   d) $\frac{1}{f} = \frac{1}{M} + \frac{1}{v}$
   b) b) $\frac{M-A}{M+D} = E$
   e) $Mgh = E - \frac{1}{2}Mv^2$
   c) c) $\frac{M}{A} + \frac{M}{B} = C$
   f) $M^2 = b^2 + \frac{c}{d}aM^2$

3. The formula for the area of the walls of a room is $A = 2h(l + w)$ where h is the height of the room, l is the length, and w the width. Calculate the

   a) area if $h = 2.3\ m,\ l = 5.3\ m,\ w = 4.5\ m$ L1
   b) area if $h = 3.41\ m,\ l = 4.6\ m,\ w = 5.83\ m$ L1
   c) height if the area is $60.42\ m^2$, the length is $4.86\ m$ and the width is $3.92\ m$. L!
   d) width if the area is $73.4\ m^2$, the height is $4.2\ m$ and the length is $6.8\ m$. L1
   e) Find the length if the area is $165.6\ m^2$, the height is $5.3\ m$ and the width is $4.8\ m$. L1

4. An aircraft flies between Rotuma and Funafuti which are about 100 km apart. The plane flies in such a way that it is always 50 km closer to Rotuma than to Funafuti, which is approximately north of Rotuma. Choosing an appropriate coordinate system find the equation that describes the path of the plane. L3

LESSON ACTIVITY 3:

SUB-STRAND	SLO NO.	SPECIFIC LEARNING OUTCOMES	SKILL LEVEL	SLO CODE	ACHIEVED
1.1	2	Solve two linear equations simultaneously. (eliminations & substitution)	1	Cal1.1.1.2	
1.2	8	Solve simultaneous equations	3	Cal1.2.3.3	
1.2	3	Solve by translating word problems into mathematical expressions.	3	Cal1.2.3.1	

SIMULTANEOUS LINEAR EQUATIONS.

Simultaneous equations are two or more equations which are true at the same time**.** **Solving a pair of simultaneous equations** means finding values for the unknown which **satisfy both** equations (make both equations true when the values are substituted).

Method 1: SOLVING SIMULTANOUS LINEAR EQUATIONS BY SUBSTITUTION.

Substitution method is when one unknown is made the subject of its equation, so that it is expressed terms of the other unknown. This expression is then substituted in the other equation.

Example A: solve $2y - 5x = 3$ ....(1)

$x + y = 5$ ....(2)

Answer:

(i) Make y the subject of the second equation (to avoid fractions)

$$y = 5 - x$$

(ii) Substitute for y into the first equation

$$2(5 - x) - 5x = 3$$

(iii) Now solve the equation (ii).

$$10 - 2x - 5x = 3$$

$$10 - 7x = 3$$

$$-7x = -7$$

$$x = 1$$

(iv) Substitute this value of x in the equation in (i).

$$y = 5 - 1$$

$$y = 4$$

Therefore the solution of $2y - 5x = 3$ and $x + y = 5$ is $x = 1$ and $y = 4$.

Method 2: SOLVING SIMULTANOUS LINEAR EQUATIONS BY ELIMINATION AND BACK SUBSTITUTION.

This involves adding together (or subtracting) the equations to eliminate an unknown. Sometimes one (or both) of the equations must be multiplied by a constant before the equations are added or subtracted, so that one unknown has (or opposite) coefficient in both equations.

Example B:

Solve the equations simultaneously: $2x + 3y = 10$ and $3x + 2y = 5$

Answer**:** The equations are numbered for easy identification.

$$2x + 3y = 10 \qquad \ldots (1)$$

$$3x + 2y = 5 \qquad \ldots (2)$$

To eliminate y, (1) is multiplied by 2 and (2) is multiplied by 3. When the resulting equations are subtracted, the 6y in each equation cancels.

$$(1) \times 2 \qquad 4x + 6y = 20 \qquad \ldots (3)$$

$$(2) \times 3 \qquad \underline{9x + 6y = 15} \qquad \ldots (4)$$

$$5x = 5 \qquad \textit{[Subtracting (4) from (3)]}$$

$$x = -1$$

The value of y is found by substituting the value $x = -1$ into either one of the original equations (1) or (2), thus:

$$2x + 3y = 10 \qquad \ldots (1)$$

$$2(-1) + 3y = 10$$

$$3y = 10 + 2$$

$$y = \frac{12}{3}$$

$$y = 4$$

The solution for the equations $2x + 3y = 10$ and $3x + 2y = 5$ is $x = -1$ and $y = 4$

[An alternative method of solution would be the elimination of x to find y. Try this method to see that the same solution is arrived at]

<u>Note</u>: *Equations should be simplified (denominators removed and like terms gathered) before any elimination occurs. If solutions involve fractions, it may be preferable to carry out two eliminations, one for each of the unknowns.*

Example C: Solve the simultaneous equations $\frac{3x-2y+7}{5} = \frac{4x+3y-11}{2}$

and $\frac{1}{2}(2x + 5y) = 8$

Answer: $\frac{3x-2y+7}{5} = \frac{4x+3y-11}{2}$

$6x - 4y + 14 = 20x + 15y - 55$ [Cross multiplication]

$14x + 19y = 69$ … (1)

And $\frac{1}{2}(2x + 5y) = 8$ ... (2)

$2x + 5y = 16$ … (2)

It is best to remove fractions when solving equations. x is now eliminated, thus:

$14x + 19y = 69$ … (1)

$(2) \times 7$ $\underline{14\text{x} + 35\text{y} = 112}$ … (3)

Subtract (3) - (1) $16y = 43$

$y = 2\frac{11}{16}$

To find x, substitute $y = 2\frac{11}{16}$ into (1) or (2):

$14x + 19y = 69$ … (1)

$14x + 19\left(2\frac{11}{16}\right) = 69$

$14x = 69 - 51\frac{1}{16}$

$14x = 17\frac{15}{16}$

$x = 1\frac{9}{32}$

The solution for the simultaneous equations $\frac{3x-2y+7}{5} = \frac{4x+3y-11}{2}$ and $\frac{1}{2}(2x + 5y) = 8$

Is: $x = 1\frac{9}{32}$ and $y = 2\frac{11}{16}$

Activity: *Substitute these values back into one of the equations to test whether they make the equation(s) true.*

Note: Some non- linear simultaneous equations can be solved in a similar way to linear simultaneous equations.

Example D: Solve the simultaneous equations $2\sqrt{x} + y^3 = 9$ and $3\sqrt{x} - 2y^3 = 10$

Answer: $2\sqrt{x} + y^3 = 9$ … (1)

$3\sqrt{x} - 2y^3 = 10$ … (2)

Multiply (1) by 2 and add (to be able to eliminate y^3)

(1) × 2 $4\sqrt{x} + 2y^3 = 18$ … (3)

$\underline{3\sqrt{x} - 2y^3 = 10}$ … (2)

$7\sqrt{x} = 28$ [adding (3) and (2) to eliminate y^3]

$\sqrt{x} = 4$ [dividing by 7]

$x = 16$

Substituting $x = 16$ into (1) gives:

$2\sqrt{16} + y^3 = 9$

$8 + y^3 = 9$ [Simplifying]

$y^3 = 1$ [rearranging]

$y = 1$

So the solutions to the simultaneous equations are $\boldsymbol{x = 16}$ and $\boldsymbol{y = 1}$

SOLVING SIMULTANEOUS LINEAR AND NON-LINEAR EQUATIONS.

To solve a linear equation and a quadratic equation simultaneously, substitute an expression for one of the variables of the linear equation into quadratic equation.

Example E:

Solve $y = x + 1$ and $y^2 - 3x = 13$ simultaeously.

Answer: substitute for y (the subject of the linear equation) in $y^2 - 3x = 13$

$(x + 1)^2 - 3x = 13$

$x^2 + 2x + 1 - 3x = 13,$ [then solve for$x$]

$x^2 - x - 12 = 0$

$(x - 4)(x + 3) = 0$

$(x - 4) = 0$ or $(x + 3) = 0$

$\therefore x = 4$ or $x = -3$

When $x = 4, y = 4 + 1 = 5$ (4,5) *[substitute into* $y = x + 1$]

When $x = -3, y = -3 + 1 = -3$ (-3, -3)

The solution also means that the line $y = x + 1$ meets or cuts the parabolic graph of the quadratic equation $y^2 - 3x = 13$ at two points (4,5) and (-3, -3)

WORDED PROBLEMS.

To form simultaneous equations from a word problem, it is important to define the variables clearly. Two equations can then be formed from the information given and solved simultaneously to evaluate the variables. The final answer should be given as a sentence that directly answers the question asked in the word problem.

Example F:

Noni and Selda together earn $132 an hour. Noni earns 20% more than Selda. What is their hourly earning?

Answer: Since hourly earnings are required, let Noni earn N dollars per hour and Selda earn S dollars per hour.

$S + N = 132$ … (1) [together they earn $132 per hour]

$N = 1.2S$ … (2) [Noni earns 120% of Selda's rate]

$S + 1.2S = 132$ [substituting for N in (1)]

$2.2S = 132$

$S = \frac{132}{2.2}$

$S = 60$

$N + 60 = 132$ [substituting in (1)]

$N = 72$

Therefore, **Selda** earns **$60** per hour and **Noni** earns **$72** per hour.

ACTIVITY 3A:

Solving Simultaneous Linear equations, non-linear equations & worded problems.

1. Find the solution(s) for each of the following pairs of simultaneous linear equations using the methods of elimination and substitution.

a) $x + 3y = 9$ and $4x + y = 14$ L1

b) $8A + 7B = 42$ and $2B - 2A = -3$ L1

c) $3y - 5x = 1$ and $4y + x = 7$ L1

d) $\frac{y-x}{2} = \frac{y+x+1}{3}$ and $2x - \frac{y}{3} = 2$ L1

2. Solve the following pairs of simultaneous equations (using the same methods as for simultaneous linear equations)

a) $\frac{1}{x} - y = 5$ and $\frac{1}{x} + y = 6$ L3

b) $\sqrt{x} + 3y^2 = 11$ and $3\sqrt{x} - y^2 = 3$ L3

c) $y = x + 7$ and $xy = -12$ L3

d) $y = 2x - 3$ and $x^2 + y^2 = 9$ L3

3. Write the simultaneous equations for each of the following problems and then solve the equations. All L3

a) The sum of two numbers is 138 and the difference is 12. Find the two numbers.

b) A pupil spends one third less time on her Maths homework than she does on her English Homework. The total time spent on both subjects is $2\frac{1}{2}$ hours. How much time does she spend on each subject?

c) A room contains 96 people. A man leaves and is replaced by a woman, leaving three times as many women as men in the room. How many men and women were there originally?

Ans: activity 3A

1- a) x=3, y=2; b) A=7/2, B=2; c) x=17/23, y=36/23; d) x=8, y=42
2- a) x=2/11, y=1/2
3- x=63, y=75

LESSON ACTIVITY 4: FACTORISATION

SUB-STRAND	SLO NO.	SPECIFIC LEARNING OUTCOMES	SKILL LEVEL	SLO CODE	ACHIEVED
1.1	5	Factorise quadratic equations	1	Cal1.1.1.5	
1.1	6	Solve quadratic equations by factorising	1	Cal1.1.1.6	
1.1	7	Solve quadratic equations by quadratic formula.	1	Cal1.1.1.7	
1.1	20	Complete the square of reducible quadratics of the form $ax^2 + bx + c$, where a = 1	1	Cal1.1.1.14	
1.1	21	Complete the square of reducible quadratics of the form $ax^2 + bx + c$, where a>1	2	Cal1.1.2.5	
1.2	4	Solve contextual problems by translating word problems into mathematical expressions.	4	Cal1.2.4.1	
1.2	17	Solve hyperbolic equations	2	Cal 1.2.2.6	
1.2	6	Analyse the existence of solutions in the context of the situation	4	Cal 1.2.4.2	

QUADRATICS FACTORISATIONS.

A quadratic function of x is one that can be written in the form $ax^2 + bx + c$. The number $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are called **coefficients** and $\boldsymbol{c}$ can also be called the **constant term**. Note that $a \neq 0$

- **Quadratics where the coefficient of x is 1**

  When the coefficient of x is one, the quadratic simplifies to $x^2 + bx + c$. Such expressions are factorised by finding two numbers which ***multiply together to give c and add together to give b***

  Example A:

  Factorise $x^2 - 2x - 15$

  Answer:

  [$= -2, c = -15$; so you need to find two numbers that will multiply to give '-15' and when you add gives '-2'; the two numbers are -5 and $+3$]

  $x^2 - 2x - 15 = (x - 5)(x + 3)$

  [Note that the two numbers in the brackets (-5, 3) have a sum of '-2' and a product of '-15']

❖ Quadratics Where the Coefficient of x is greater than 1

There are two cases:

i. **The quadratics has a common factor.**
The common factor is taken out first and the remaining quadratics can be factorised

Example B: Factorise $8x^2 - 8x - 48$

Answer: $8x^2 - 8x - 48 = 8(x^2 - x - 6) = 8(x-3)(x+2)$

ii. **The quadratics has no common factor.**

Example C: Factorise $4x^2 + 7x - 15$

Step 1: Multiply the coefficient of x^2 (4) and the constant term (-15) together.

$$4 \times (-15) = -60$$

Step 2: Now look for the pair of factors of -60 whose sum is equal to the coefficient of x(which is 7).

The number is -5 and 12. Using these two numbers, the middle term is expressed as a sum of two terms.

$4x^2 + 7x - 15 = 4x^2 - 5x + 12x - 15$

$= x(4x-5) + 3(4x-5)$ [the term in the brackets must be the same or common,for factorisation in the next step.]

$= (x+3)(4x-5)$

❖ The Difference of Two Squares

Any expression that can be written generally as $a^2 - b^2$ can be factorised and written using the formula : $\boldsymbol{a^2 - b^2 = (a+b)(a-b)}$

Example D: Factorise $1 - 9x^2$

Answer: $1 - 9x^2 = (1+3x)(1-3x)$

Example E: Factorise $128 - 18a^4$

Answer: $2(64 - 9a^4) = 2((8)^2 - (3a^2)^2) = 2(8+3a^2)(8-3a^2)$

SOLVING QUADRATIC EQUATIONS.

A quadratic equation is an equation which is written in the form $ax^2 + bx + c = 0$

- **Solving Quadratics by Factorising.**

  In some cases quadratic equations can be solved by factorising. This relies on the fact that if the product of two factors is zero, then one or other of the factor must be zero.

  Example F: Solve the quadratic equation $x^2 - x = 6$

  Answer: $x^2 - x = 6$ [re-arrange the equation so it is equal to zero]

  $x^2 - x - 6 = 0$

  $(x - 3)(x + 2) = 0$

  $(x - 3) = 0 \; or \; (x + 2) = 0$

  $x = 3$ or $x = -2$

- **Solving Quadratics by using Quadratic Formula.**

  Sometimes a quadratic equation has a solution, yet cannot be factorised readily. In these case the quadratic formula is used.

  The solution to the general quadratic equation, $ax^2 + bx + c = 0$

$$\boxed{\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}}$$

  The solutions of an equation are often called the **roots** of the equation.

Example G: Find the solutions to the equation $2x^2 + 8x - 3 = 0$, correct to 2 decimal places.

Answer: $2x^2 + 8x - 3 = 0$ gives $a = 2, b = 8, c = -3$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-8 \pm \sqrt{8^2 - 4 \times 2 \times (-3)}}{4}$

$x = \frac{-8 \pm \sqrt{88}}{4}$

$x = \frac{-8 + \sqrt{88}}{4} \; or \; x = \frac{-8 - \sqrt{88}}{4}$

$x = 0.35 \; or \; x = -4.35$ (2dp)

- **Quadratics in Completed Square Form**

Any quadratics can be expresses in the form of a perfect square plus or minus a constant, using a process called **completing the square.** This process of completing the square is explained in the following examples:

Example H: Express $x^2 + 3x - 5$ in the completed square form $(x-p)^2 + q$

Answer: Using the rule: $(\frac{1}{2} coefficient\ of\ x)^2 = \left(\frac{3}{2}\right)^2 = 2\frac{1}{4}$

Arrange the quadratic in perfect square form.

$$x^2 + 3x - 5 = \left(x^2 + 3x + 2\tfrac{1}{4}\right) - 2\tfrac{1}{4} - 5$$

$$= \left(x + \tfrac{3}{2}\right)^2 - 7\tfrac{1}{4}$$

Note: If the coefficient of x^2 is not 1, use the coefficient of x^2 as a common factor as follows.

Example I: Express $6x^2 - 12x + 5$ in the form $6(x-p)^2 + q$

Answer:

$$6x^2 - 12x + 5 = 6\left(x^2 - 2x + \tfrac{5}{6}\right)$$

$$= 6\left[(x^2 - 2x + 1) - 1 + \tfrac{5}{6}\right] \quad \text{[completing the square]}$$

$$= 6\left[(x-1)^2 - \tfrac{1}{6}\right]$$

$$= 6(x-1)^2 - 1 \quad \text{[expanding]}$$

- **Solving Quadratic Equation using Completing the Square.**

Another method used to solve quadratic equations which cannot be factorised is completing the square. Firstly transfer the constant term to the RHS then divide through by the coefficient of x^2.

Example J: Solve $2x^2 - 3x - 7 = 0$

Answer:

$$2x^2 - 3x = 7$$

$$x^2 - \tfrac{3}{2}x = \tfrac{7}{2}$$

$$\left(\tfrac{1}{2} coefficient\ of\ x\right)^2 = \left(\tfrac{1}{2} \times -\tfrac{3}{2}\right)^2 = \left(-\tfrac{3}{4}\right)^2 = \left(\tfrac{3}{4}\right)^2 \ or\ \tfrac{9}{16}$$

$$x^2 - \tfrac{3}{2}x + \left(\tfrac{3}{4}\right)^2 = \tfrac{7}{2} + \tfrac{9}{16}$$

$$\left(x - \tfrac{3}{4}\right)^2 = \frac{56+9}{16}$$

$$x - \tfrac{3}{4} = \pm\frac{\sqrt{65}}{4}$$

$$x = \frac{3 \pm \sqrt{65}}{4}$$

$$x = 2.77 \; or \; x = -1.27$$

- **Word Problems [Applications of Quadratics equations]**

Example K: The length of a small room is 3 m more than its width. The room has an area of 72 m^2.Find the width of the room.

Answer: Let the width of the room be w

$\therefore$ the length is $w + 3$

$\therefore w(w + 3) = 72$

$w^2 + 3w = 72$

$w^2 + 3w - 72 = 0 \qquad [a = 1, b = 3, c = -72]$

Use quadratic formula:

$$x = \frac{-3 \pm \sqrt{(-3)^2 - 4 \times 1 \times (-72)}}{2}$$

$$x = \frac{-3 \pm \sqrt{297}}{2}$$

$$x = \frac{-3 + \sqrt{297}}{2} \; or \; x = \frac{-3 - \sqrt{297}}{2}$$

$$x = 7.12 \; or \; x = -10.12$$

Therefore, the width is 7.12 m - reject -10.12 because it is impossible to have a negative length.

ACTIVITY 4A**:** Factorising/ solving Quadratics equations and Applications

1. Factorise these quadratic expressions. L1
   a) $3x^2 - 6x - 24$
   b) $7x^2 - 23x + 6$
   c) $e^x - e^x sin^2(\theta)$
   d) $36x^4 - \frac{1}{4}$

2. Solve these quadratic equations by factorising. L1
   a) $x^2 + 5x - 36 = 0$
   b) $2x^2 - 15x + 28 = 0$
   c) $4x^2 = 9$

3. Solve these quadratic equations using quadratic formula. Give answers correct to 2dp. L1

a) $x^2 = 2 - 5x$
b) $2x^2 + 14x + 3 = 0$
c) $(3x - 1)(x - 9) = (x + 2)(x - 4)$

4. Solve these quadratics equations by completing the square. L2
   a) $x^2 - 2x - 5 = 0$
   b) $2x^2 + 8x - 17 = 0$
   c) $5x^2 + 3x - 2 = 0$

5. Rewrite each of the following in the form $(x + p)^2 + q$ L2
   a) $x^2 + 2x + 3$ b) $x^2 - 5x$

6. Rewrite each of the following in the form $a(x + p)^2 + q$ L2
   a) $3x^2 + 6x - 4$ b) $5x^2 + 4x + 3$

7. A factory has daily overhead costs of \$2000 while each item produced costs \$100. What is the cost of producing 1000 items? L2

8. The difference between two positive numbers is 3. The difference between their reciprocals is $\frac{1}{90}$. What are the two numbers? L3

9. Tara has a rectangular lawn that is 11 m long and 8 m wide. The lawn is to be surrounded by the path. The width of the path on each side of the lawn is the same. The total area of the lawn and the path is $100\ m^2$. What are the dimensions of the path? L3
10. Show that $x^2 + 4x + 3 = 0$ has no real solution L3
11. Show whether each of these quadratic equations has unequal real solutions, a repeated solution, or no real solution: (L3)
    a) $x^2 - 5x + 7 = 0$ b) $x^2 + 5x + 7 = 0$ c) $9x^2 - 12x + 4 = 0$
12. Since the beginning of the month a reservoir has been losing water at a constant rate. On the 10^{th} of the month the water in the reservoir is 300 million gallons, and on the 18^{th} only 262 million gallons. How much water is in the reservoir on the 14^{th} of the month? L4 *[Mizrahi & Sullivan "Mathematics: An applied Approach"]*

Ans: Activity 4A

1- *a) 3(x+2)(x-4); b) (7x-2)(x-3); d)* $(6x^2 + 1/2)(6x^2 - 1/2)$

2- *a) x=4,x=-9; b) x=4,x= 7/2; c) x=3/2, x=-3/2*

3- *a) x=0.37,x= -5.38; b) x=-0.22, x=-6.79; c) x=12.31, x=0.69*

4- *a)* $x = 1 + \sqrt{6}; x = 1 - \sqrt{6}$

LESSON ACTIVITY 5: Remainder and factor theorem

SUB-STRAND	SLO NO.	SPECIFIC LEARNING OUTCOMES	SKILL LEVEL	SLO CODE	ACHIEVED
1.1	15	Divide a polynomial by $(x+a)$	1	Cal1.1.1.11	
1.1	16	Find the remainder to a function for $x = a$ where $(x-a)$ is not a factor.	1	Cal1.1.1.12	
1.1	8	Factorise a cubic function using factor theorem.	2	Cal1.1.2.2	
1.2	18	Use Remainder and Factor theorems involving straightforward substitution method and long division.	2	Cal1.2.2.5	
1.2	19	Use Remainder and Factor theorems to completely factories a polynomial of degree 3.	3	Cal1.2.3.7	
1.2	20	Use Remainder and Factor theorems to find unknowns in a polynomial.	4	Cal1.2.4.3	

POLYNOMIALS

Any polynomial can be expressed in the form:

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0.$$

The number $a_n, a_{n-1}, \ldots\ a_2, a_1$ and a_0 are called **coefficients.**

The **degree** of the polynomial is the highest power of x present.

❖ **Synthetic Division**

A method similar to long division algorithm is used to express this fraction in quotient plus remainder form.

Example A: Write $\frac{x^2-3x+4}{x+9}$ in Quotient plus Remainder form.

Answer:

$$\begin{array}{r} x-12 \\ (x+9)\overline{)\,x^2-3x+4} \\ \underline{x^2+9x} \\ -12x+4 \\ \underline{-12x-108} \\ 112 \end{array}$$

[remainder]

Solution: $\frac{x^2-3x+4}{x+9} = \boldsymbol{x - 12 + \frac{112}{x+9}}$

ACTIVITY 5A:

1. Write each of the following in quotient plus remainder form. L1

   a) $\frac{x^3+2x^2-5x+6}{x+4}$ b) $\frac{x^3+x-2}{x^2+4}$

2. Determine a and b such that

   $(x^2 + ax + b)(x^2 + 3x - 2) = x^4 + 3x^3 + 3x^2 + 15x - 10$ L2

❖ **The Remainder Theorem.**

The remainder theorem states: when a polynomial $p(x)$ is divided by a *linear* factor $(x + a)$ the remainder is $p(-a)$. Similarly, division by $(x - a)$ results in a remainder of $p(a)$.

The result can be extended so that in general, division of $p(x)$ by $(ax + b)$ yields a remainder of $p\left(\frac{-b}{a}\right)$

EXAMPLE A: Calculate the remainder when $x^2 - 3x + 4$ is divided by $(x - 9)$

Answer: For division by $(x - 9)$ we evaluate $p(+9)$

$$p(x) = x^2 - 3x + 4$$
$$p(-9) = (-9)^2 - 3 \times -9 + 4$$
$$= 81 + 27 + 4$$
$$= 112$$

ACTIVITY 5B:

1. What is the remainder when $x^3 + 4x^2 - 7x + 10$ is divided by $(x - 2)$? L1
2. What is the remainder when $8x^3 + 4x^2 - 16x + 12$ is divided by $(x + 1)$ L1
3. What is the remainder when $2x^3 + 3x^2 - 7x - 1$ is divided by $(2x - 1)$ L1
4. Determine the value of b if the remainder is 17 when the polynomial $p(x) = x^4 - x^3 + bx^2 - 2x + 1$ is divided by $(x - 2)$ L2
5. What is the value of ‘m’ if division of $p(x) = x^4 - mx^3 + x^2 - 2x + 1$ by (x + 2) yields a remainder of 9? L2
6. What conclusion can be made if, when a polynomial $p(x)$ is divided by $(x - 3)$, the remainder is zero? L1

THE FACTOR THEOREM

The factor theorem is used to factorise polynomials. The factor theorem in its simplest form, states that $(x - a)$ will be a factor of any polynomial $p(x)$ if and only if $p(a) = 0$. This implies that:

- $(x - a)$ is a factor of $p(x)$ means that $p(a) = 0$
- $(x + a)$ is a factor of $p(x)$ means that $p(-a) = 0$
- $(bx - a)$ is a factor of $p(x)$ means that $p\left(\frac{a}{b}\right) = 0$

The converse is true that if p(a) = 0 then (x - a) is a factor of p(x)

Example A: Find the factors of $p(x) = x^3 - 4x^2 - 7x + 10$

Answer:

This expression is a cubic and therefore has three roots. Each root is a factor of 10 – the constant term that results when the cubic is expanded.

Possible numbers to use [substitute into p(x)] in the factor theorem process are $\pm 1, \pm 2, \pm 5\ \ and\ \ \pm 10$. These numbers are factors of 10.

$p(1) = 1^3 - 4(1)^2 - 7(1) + 10 = 1 - 4 - 7 + 10 = 0$
$\therefore (x - 1)\ is\ a\ factor$

$p(-1) = (-1)^3 - 4(-1)^2 - 7(-1) + 10 = -1 - 4 + 7 + 10 = 12 \neq 0$

$\therefore (x + 1)\ is\ not\ a\ factor$

$p(-2) = 1 = (-2)^3 - 4(-2)^2 - 7(-2) + 10 = -8 - 16 + 14 + 10 = 0$

$\therefore (x + 1)\ is\ \ a\ factor$

We have found two factors: $(x - 1)\ and\ (x + 2)$

The final factor can be obtained by inspection:

$$x^3 - 4x^2 - 7x + 10 = (x - 1)(x + 2)(x + k)$$

Equate the constant term: $10 = -1 \times 2 \times k$

$10 = -2k$

$-5 = k$

Therefore, the third factor is $(x - 5)$

The full factorising is $x^3 - 4x^2 - 7x + 10 = (x - 1)(x + 2)(x - 5)$

NOTE: Long division and factor theorem can be used together to factorise or solve a cubic equation. One factor of the cubic is found by trial and error using factor theorem and long division yields a quadratic expression which can be factorised (if possible). If the quadratic equation does not factorise it can be solved using the quadratic formula.

Example B: Solve $4x^3 - 13x + 6 = 0$

Answer: Let $p(x) = 4x^3 - 13x + 6$

[substitute values of $x = \pm 1, \pm 2, \pm 3, \pm 6$]

$p(-2) = 4(-2)^3 - 13(-2) + 6$

$p(-2) = 0$

$\therefore (x+2)\ is\ a\ factor$

Long division will give the other factor:

$$
\begin{array}{r l}
 & 4x^2 - 8x + 3 \\
x+2 \big) & 4x^3 + 0x^2 - 13x + 6 \qquad [0x^2 term\ is\ inserted] \\
 & \underline{4x^3 + 8x^2} \\
 & \quad -8x^2 - 13x \\
 & \quad \underline{-8x^2 - 16x} \\
 & \qquad\qquad 3x + 6 \\
 & \qquad\quad - \underline{(3x + 6)} \\
 & \qquad\qquad 0 \ + \ 0
\end{array}
$$

Thus $4x^3 - 13x + 6 = (x+2)(4x^2 - 8x + 3)$

Factorising the quadratic gives

$p(x) = (x+2)(2x-1)(2x-3)$

$\therefore 4x^3 - 13x + 6 = 0$

$(x+2)(2x-1)(2x-3) = 0$

Solutions are: $x = -2, x = \frac{1}{2}, x = \frac{3}{2}$

ACTIVITY 5C:

1. a) Show that $(x - 3)$ is a factor of $p(x) = x^3 - 2x^2 - 9x + 18$ L1

   b) Hence solve $x^3 - 2x^2 - 9x + 18 = 0$ L2

2. Fully factorise $p(x) = 3x^2 + 8x^2 - 7x - 12$ L2

3. If $(x - q)$ is a factor of $x^3 + 2x^2 - 9x - 18$ determine the possible values of q. L3

4. Determine the remaining factor and the value of p and q if $(2x - 5)$ and $x - 1)$ are both factor of $2x^3 + 3px^2 + 11qx - 20$ L3

5. Solve $4x^3 - 12x^2 + 9x - 2 = 0$ L4

Ans:

LESSON ACTIVITY 6: Indices and Logarithms

SUB-STRAND	SLO NO.	SPECIFIC LEARNING OUTCOMES	SKILL LEVEL	SLO CODE	ACHIEVED
1.1	9	Apply laws of indices to simplify exponential expressions.	1	Cal1.1.1.7	
1.1	10	Solve straightforward exponential equations.	1	Cal1.1.1.8	
1.1	11	Apply laws of logarithms to simplify logarithmic expression	1	Cal1.1.1.9	
1.1	12	Solve straight forward logarithmic equations.	1	Cal1.1.1.10	
1.2	12	Solve logarithmic equations involving single logs, and addition and subtraction of logs	2	Cal1.2.2.3	
1.2	13	Solve logarithmic equations involving logs rules on powers and coefficients	3	Cal1.2.3.4	
1.2	5	Analyse the existence of solutions in the context of a simple situation.	3	Cal1.2.3.2	
1.2	6	Analyse the existence of solutions in the context of the complex situation.	4	Cal1.2.4.2	
1.2	9	Solve exponential equations involving negative powers with positive exponents	1	Cal1.2.1.3	
1.2	10	Solve exponential equations including application of laws of indices	1	Cal1.2.1.4	
1.2	11	Solve exponential equations involving same base but different powers	2	Cal1.2.2.2	

INDICES

Indices involves expression of the form x^n where x is the **base** and n is the **powe**r or **index** or **exponent**. Some important properties of indices are:

$x^m \times x^n = x^{m+n}$	$x^{-n} = \frac{1}{x^n}$
$x^m \div x^n = x^{m-n}$	$\left(\frac{x}{y}\right)^{-m} = \left(\frac{y}{x}\right)^m = \frac{y^m}{x^m}$
$(x^m)^n = x^{mn}$	$x^{\frac{1}{y}} = \sqrt[y]{x}$
$x^0 = 1$	$x^{\frac{m}{n}} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$

$\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$	

Example A: Simplify $\frac{15x^{-3}y^2}{(3x^{-2}y^4)^2}$

Answer: $\frac{15x^{-3}y^2}{(3x^{-2}y^4)^2} = \frac{15x^{-3}y^2}{3^2(x^{-2})^2(y^4)^2} = \frac{15x^{-3}y^2}{9x^{-4}y^8} = \frac{5x^1y^{-6}}{3} = \frac{5x}{3y^6}$

- **Solving Exponential Equations with the Same Base**

Exponential equation which can be written as the same base, the power or index can be equated to solve to find the unknown variable.

Example B: Solve $9^{2x-5} = 27$

Answer: $(3^2)^{2x-5} = 3^3$

$3^{4x-10} = 3^3$

$4x - 10 = 3$

$x = \frac{13}{4}$

- **Solving Exponential Equations with the Different Base**

Exponential equations with a different base and cannot be written as a same base and having unknown in the exponent(s) can be solve using logarithms.

Example C: Solve $3^{2x-1} = 5$

Answer: Take ln on both sides

$3^{2x-1} = 5$

$ln(3^{2x-1}) = \ln 5$

$(2x-1)\ln 3 = \ln 5$

$(2x-1) = \frac{\ln 5}{\ln 3}$

$x = \left(\frac{\ln 5}{\ln 3} + 1\right) \div 2$

$x = 1.2325$ (4dp)

Note: In more complicated equations where x appears more than once, collect like terms and simplify as necessary.

Example D: Solve $10^{3x-2} = 5^{x+1}$

Answer: Take log on both sides

$$10^{3x-2} = 5^{x+1}$$

$$log(10^{3x-2}) = log(5^{x+1})$$

$$(3x-2)\log 10 = (x+1)\log 5$$

$$3xlog\ 10 - 2\log 10 = xlog\ 5 + \log 5$$

$$x(3\log 10 - \log 5) = \log 5 + 2\log 10$$

$$x = \frac{log5+2log10}{3log10-log5} = \frac{\log(5\times10^2)}{log(10^3\div5)}$$

$$x = 1.173 \qquad \text{(4sf)}$$

LOGARITHMS

- **Formal definition of Logarithm.**

$$b^p = q \quad \leftrightarrow \quad \log_b(q) = p$$

$$\begin{matrix} \uparrow & \uparrow \\ index\ form & log\ form \end{matrix}$$

Where b is the **base**, p is called the **logarithm** and a is the **number.**

Example A: Solve $\log_{10}(2x+3) = 2.5$

Answer: By the identity above, the equation becomes

$$2x + 3 = 10^{2.5}$$

$$x = \frac{10^{2.5} - 3}{2}$$

$$x = 156.6 \qquad \text{(4sf)}$$

- **Properties of Logarithms.**

$$\log_b x + \log_b y = \log_b(xy)$$

$$\log_b x - \log_b y = \log_b\left(\frac{x}{y}\right)$$

$$\log_b(x^n) = n\log_b x$$

$$if\ \ \log_b x = \log_b y\ \ then\ x = y$$

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Example B: Simplify $2\log(5) + 2\log(10)$

Answer: $2\log(5) + 2\log(10) = \log(5^2) + \log(10^2)$

$$= \log(25) + \log(100)$$

$$= \log(25 \times 100)$$

$$= \log(2500)$$

Example C: Solve $\log_{10}(x+3) - \log_{10}(x) = 2$

Answer:

$$\log_{10}(x+3) - \log_{10}(x) = 2$$

$$\log_{10}\left(\frac{x+3}{x}\right) = 2$$

$$\left(\frac{x+3}{x}\right) = 10^2$$

$$(x+3) = 100x$$

$$x = \frac{3}{99} = \frac{1}{33}$$

ACTIVITY 7A:

1. Simplify the following:
   a) $(16x^4)^{\frac{-3}{4}}$ L1

   b) $\frac{6m^5}{\sqrt{9m^{16}}}$ L1

   c) $\log_2(6) + \log_2(8) - \log_2(3)$ L1

   d) $5\log(2) - \frac{1}{2}\log(16)$ L1

2. Solve the following for x:

   a) $\log_x 64 = 3$ L1

   b) $2\log(x) = \log(2x) + \log(3)$ L2

   c) $\log_2(x+5) - \log_2 3 = \log_3(x-3)$ L2

   d) If $\ln x_o = 1$ and $\ln x = 2$ find an expression for $'x'$ in terms of x_o. L2

e) $4^{3x+5} = 8^{4x-3}$ L2

f) $5^{3x-2} = 300$ L2

g) $7^{2x+1} = 3^{4x}$ L2

h) $27^{x} = 3^{x+6}$ L2

i) $6^{2x+1} = 8^{x+p}$ L3

3. A newborn giraffe is 1.8 metres tall. A formula that gives the height H metres of the giraffe over the first five years is $\boldsymbol{H = 1.8 \times 3^{0.16t}}$ where t is the time in years since the giraffe was born. How long does it take for the giraffe to reach a height of 2.7 metres? L3

   A student learns to type y words per minute after t days of practice. The relationship between y and t is given by: $y = 120(1 - e^{-0.15t})$. How many days does it take the student to learn to type at 100 words per minute? L3

4. The total annual water consumption per person W(y) in the Pacific Islands is estimated to be $W(y) = 1200(1 + e^{0.5(y-2000)})$ where 'y ' is the year and W(y) is the total consumption in gallons in that year .

   (a) What was the total annual water consumption in the year 2000? L2

   (b) What will the total water consumption be in 2020? L3

   [*Mizrahi & Sullivan "Mathematics: An Applied Approach*"]

LESSON ACTIVITY 7: [PARTIAL FRACTIONS]

SUB-STRAND	SLO NO.	SPECIFIC LEARNING OUTCOMES	SKILL LEVEL	SLO CODE	ACHIEVED
1.1	13	Express a single algebraic fraction as a sum of partial fractions where denominators are linear or repeated linear.	2	Cal1.1.2.3	
1.1	14	Express a single algebraic fraction as a sum of its partial fractions where denominators of fractions include a non-linear function or an irreducible quadratic function	3	Cal1.1.3.1	

PARTIAL FRACTIONS

The decomposition of a fractional algebraic expression is the process that involves expressing a rational algebraic expression as a sum of its parts, i.e, the sum of two partial fractions.

Composition

Example A:

$$\frac{2}{x+1}+\frac{5}{x-3}=\frac{2(x-3)(5(x+1)}{(x+1)x-3)}=\frac{7x-1}{(x+1)(x-3)}$$

Decomposition

Before attempting problems, ensure that the fraction is **bottom heavy**, i.e. the degree of the numerator is less than the degree of the denominator.

Note: 1. If the fraction is not bottom heavy, **do a long division first.**
2. Be sure the **denominator is factorized.**

TYPE 1: DENOMINATOR WITH ONLY LINEAR FACTORS

For the case of linear factors, we use the fact that corresponding to each linear factor $ax+b$, occurring once in the denominator, that there will be a partial fraction of the form $\frac{A}{ax+b}$, where A is a constant to be determined.

Example B***:*** Express $\frac{2x+1}{(x-1)(x+2)}$ as a sum of partial fractions.

$$\frac{2x+1}{(x-1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)}$$

Now multiply with the common denominator; $(x-1)(x+2)$ to clear out the fractions

$$2x+1 = A(x+2) + B(x-1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ equation 1}$$

There are two methods to use: **Substitution method** and **Coefficient method**.

SUBSTITUTION METHOD

Equating the two factors in the denominator to equal zero and solving gives the values of x =-2 and x = 1. Substitute the values of x into equation 1 to find the constants A and B.

$$x = 2: \begin{aligned} &2(-2)+1 = A(-2+2) + B(-2-1) \\ &-4+1 = A(0) + B(-3) \\ &-3 = -3B \\ &\therefore B = 1 \end{aligned}$$

$$\text{For } x = 1: \begin{aligned} &2(1)+1 = A(1+2) + B(1-1) \\ &3+1 = A(3) + B(0) \\ &3 = 3A \\ &\therefore A = 1 \end{aligned}$$

Therefore $\dfrac{2x+1}{(x-1)(x+2)} = \dfrac{1}{(x-1)} + \dfrac{1}{(x+2)}$

EQUATING COEFFICIENTS METHOD

We can equate coefficients:

$$2x+1 = A(x+2) + B(x-1)$$

$$= Ax + 2A + Bx - B \qquad \textit{expand RHS}$$

$$= Ax + Bx + 2A - B \qquad \textit{re-arranging and collecting like terms}$$

$$2x+1 = (A+B)x + 2A - B \qquad \textit{factorizing}$$

Equating to the coefficients of the expression $2x+1$

For coefficient of x: $A + B = 2$ ……… equation 1

For constant terms: $2A - B = 1$………equation 2

We have two simultaneous equations to solve and that's:

$$+\begin{aligned} A + B &= 2 \\ 2A - B &= 1 \end{aligned}$$

$$3A = 3$$
$$\therefore A = 1$$

Substitute $A = 1$ into $A + B = 2$ and solve for B;

$$1 + B = 2$$
$$\therefore B = 1$$

Therefore $$\frac{2x+1}{(x-1)(x+2)} = \frac{1}{(x-1)} + \frac{1}{(x+2)}$$

Example C**:** Express $\dfrac{x^3 + 2x - 4}{x^3 - 4x}$ as a sum of partial fractions.

Note: The fraction is not bottom heavy so do a long division and factorize the denominator after dividing.

$$\begin{array}{r} 1 \\ x^3 - 4x \,\big)\, \overline{x^3 + 2x - 4} \\ -\; x^3 - 4x \\ 6x - 4 \end{array}$$

$$\therefore \frac{x^3 + 2x - 4}{x^3 - 4x} = 1 + \frac{6x - 4}{x^3 - 4x}$$

$$\frac{6x-4}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x+2)}$$

Multiplying both sides of the equation by the common denominator $x(x-2)(x+2)$ to clear out the fractions gives:

$$6x - 4 = A(x-2)(x+2) + Bx(x+2) + C(x)(x-2)$$

Using the substitution method, let $x = -2,\ 0$ and 2.

For $x = -2$: $6(-2) - 4 = A(-2-2)(-2+2) + B(-2)(-2+2) + C(-2)(-2-2)$

$$-12 - 4 = A(-4)(0) + B(-2)(0) + C(-2)(-4)$$

$$-16 = A(0) + B(0) + 8C$$

$$-16 = 8C$$

$$\therefore C = -2$$

For $x = 0$: $6(0) - 4 = A(0-2)(0+2) + B(0)(0+2) + C(0)(0-2)$

$$-4 = A(-2)(2) + B(0)(2) + C(0)(-2)$$

$$-4 = -4A + B(0) + C(0)$$

$$-4 = -4A$$

$$\therefore A = 1$$

For $x = 2$: $\quad 6(2) - 4 = A(2-2)(2+2) + B(2)(2+2) + C(2)(2-2)$

$$12 - 4 = A(0)(4) + B(2)(4) + C(2)(0)$$

$$8 = A(0) + 8B + C(0)$$

$$8 = 8B$$

$$\therefore B = 1$$

Hence, $\quad \dfrac{x^3 + 2x - 4}{x^3 - 4x} = 1 + \dfrac{A}{x} + \dfrac{B}{(x-2)} + \dfrac{C}{(x+2)}$

$$= 1 + \frac{1}{x} + \frac{1}{(x-2)} - \frac{2}{(x+20}$$

ACTIVITY 7A

Express each of the following as a sum of partial fractions

a. $\dfrac{x+3}{(x+1)(x+2)}$ L2

b. $\dfrac{2x^4 - x^3 - 9x^2 + x - 12}{x^3 - x^2 - 6x}$ L2

c. $\dfrac{x^3}{x^2 + 3x + 2}$ L2

d) $\dfrac{x^2 + 3}{x^2 + 3x}$ L2

e) $\dfrac{2x + 4}{3x^2 + 5x + 2}$ L2

f) $\dfrac{x - 1}{4x^2 + x}$ L2

Ans: Activity 7A

a) $\frac{3}{x+1} - \frac{2}{x+2}$

b) $2x + 1 + \frac{1}{x} - \frac{1}{x+2} + \frac{3}{x-3}$

d) $1 + \frac{1}{x} - \frac{4}{x+3}$

TYPE 2: DENOMINATOR WITH REPEATED LINEAR FACTORS

For the case of repeated linear factor, we use the fact that corresponding to each linear factor $ax+b$ that occurs n times, in the denominator, there will be n partial fractions.

$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)}+\ldots+\frac{A_n}{(ax+b)^n}$ where $A_1, A_2, \ldots A_n$ are constants to be determined.

Example D:

Express $\frac{x+4}{(x+3)(x+2)^2}$ as a sum of partial fractions

Single linear factor Repeated linear factor

For this type we do the following: $\frac{x+4}{(x+3)(x+2)^2}=\frac{A}{(x+3)}+\frac{B}{(x+2)}+\frac{C}{(x+2)^2}$

Note: Each power of the factor(x+2) has a separate function.

Multiply both sides of the equation by the common denominator $(x+3)(x+2)^2$ gives:

$x+4=A(x+2)^2+B(x+3)(x+2)+C(x+3)$ ……………. Equation 1

Using substitution method let the values of $x=-2$ and -3.

For $x=-2$: $-2+4=A(-2+2)^2+B(-2+3)(-2+2)+C(-2+3)$

$$2=A(0)+B(1)(0)+C(1)$$

$$\therefore C=2$$

For $x=-3$: $-3+4=A(-3+2)^2+B(-3+3)(-3+2)+C(-3+3)$

$$1=A(1)^2+B(0)(-1)+C(0)$$

$$\therefore A=1$$

Since the is no other value to make a factor in equation 1 zero, we can choose some other values of x. Let $x=0$. (since it is an identity). Use the value of C = 2 and A = 1 that's been calculated.

$$x+4=A(x+2)^2+B(x+3)(x+2)+C(x+3)$$

$$0+4=1(0+2)^2+B(0+3)(0+2)+2(0+3)$$

$$4=4+B(3)(2)+2(3)$$

$$4=4+6+6B$$

$$4-10=6B$$

$$-6=6B$$

$$\therefore B=-1$$

Therefore $\frac{x+4}{(x+3)(x+2)^2}=\frac{1}{x+3}-\frac{1}{(x+2)}+\frac{2}{(x+2)^2}$

ACTIVITY 7B

Express each of the following as sum of partial fractions

L2

L3

L2

a) $\frac{x}{(x+1)^2}$ L2

b) $\frac{x}{(x-1)^2(x-2)}$ L2

c) $\frac{1}{x^3-x^2}$ L3

d) $\frac{7x-4}{(x-1)^2(x+2)}$

e) $\frac{x-8}{x^3-4x^2+4x}$

f) $\frac{1}{x(x+3)^2}$

Ans:

a) $\frac{1}{x+1}-\frac{1}{(x+1)^2}$

d) $-\frac{2}{x+2}+\frac{2}{x-1}+\frac{1}{(x-1)^2}$

TYPE 3: DENOMINATOR WITH QUADRATIC FACTOR

For the case of quadratic factors, we use the fact that corresponding to each irreducible (cannot be further factored) quadratic factor of the form ax^2+bx+c that occurs once in the denominator, there is a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$ where A and B are constants to be determined.

<u>Example E</u>*:* Express $\frac{x^2+x-2}{(x+1)(x^2+1)}$ as s sum of partial fractions.

Answer: $\frac{x^2+x-2}{(x+1)(x^2+1)}=\frac{A}{(x+1)}+\frac{Bx+C}{(x^2+1)}$

Multiplying both sides of the equation with the common denominator gives:

$x^2+x-2=A(x^2+1)+(Bx+C)(x+1)$

Using equating coefficient method:

$x^2+x-2=Ax^2+A+Bx^2+Bx+Cx+C$ expanding brackets

$x^2+x-2=(A+B)x^2+(B+C)x+A+C$ collecting like-terms and factorizing

Equating the coefficients of:

x^2 : $A+B=1$ ………………. **equation 1**

x : $B+C=1$ ……………........**equation 2**

Constant term: $A+C=-2$ ……………..**equation 3**

Make B the subject in **equation 2**: $B=1-C$ then substitute it into **equation 1**.

$A+1-C=1$

$A-C=0$ ……………….**equation 4**

Solve **equation 3** and **4** simultaneously.

$$\begin{array}{r} A+C=-2 \\ -\quad \underline{A-C=0} \\ 2A=-2 \\ \therefore A=-1 \end{array}$$

Substitute $A = -1$ into **equation 4** to find C.

$-1 - C = 0$

$\therefore C = -1$

Substitute $C = -1$ into **equation 2** to find B.

$B + -1 = 1$

$\therefore B = 2$

Therefore $\dfrac{x^2 + x - 2}{(x+1)(x^2+1)} = \dfrac{-1}{(x+1)} + \dfrac{2x-1}{(x^2+1)}$

Try using substitution method to see if you will get the same answer!

ACTIVITY 7C

Express each of the following as sum of partial fractions

a) $\dfrac{x^4 + x^2 + 1}{x(x^2+1)}$ L3

b) $\dfrac{x^2 + 5x + 2}{(x+1)(x^2+1)}$ L3

c) $\dfrac{3x^2 + 1}{(x+1)(x^2+1)}$ L3

d) $\dfrac{4x + 4}{x^3 + 4x}$ L3

e) $\dfrac{x^2 + x + 5}{(x+1)(x^2+4)}$ L3

f) $\dfrac{sx}{(x+2)(x^2+1)}$ L3

Ans:

a) $x + \frac{1}{x} - \frac{x}{x^2+1}$

b) $\frac{2x+3}{x^2+1} - \frac{1}{x+1}$

TYPE 4: DENOMINATOR WITH REPEATED QUADRATIC FACTORS

For the case of repeated quadratic factors, we use the fact that corresponding to each irreducible quadratic factor ax^2+bx+c that occurs n times in the denominator there will be n partial fractions.

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

where A_1, A_2, … A_n, B_1, B_2, … B_n are constants to be determined.

<u>Example 1</u>: Express $\dfrac{2x+x}{(x^2+1)^2}$ as a sum of partial fractions.

We write similar fractions as for repeated linear factors except we put terms like $Ax+B$ in each numerator as shown below:

$$\frac{2x+x}{(x^2+1)^2}=\frac{Ax+B}{(x^2+1)}+\frac{Cx+D}{(x^2+1)^2}$$

We proceed as before and use equating the coefficient method:

$2x^2+x=(Ax+B)(x^2+1)+Cx+D$

$=Ax^3+Ax+Bx^2+B+Cx+D$ expanding

$=Ax^3+Bx^2+Ax+Bx+B+D$ rearranging

$=Ax^3+Bx^2+(A+C)x+B+D$ collecting like-terms and factorizing

Now equating coefficients of powers of x gives us:

For x^3 : $A=0$

For x^2 : $B=2$

For x : $A+C=1$

$0+C=1$ substituting for A

$\therefore C=1$

For constants terms x^0 : $B+D=0$

$2+D=0$ substituting for B

$\therefore D=-2$

Hence $=\dfrac{0x+2}{(x^2+1)}+\dfrac{x-2}{(x^2+1)^2}$

ACTIVITY 7D

Express the following as sums of partial fraction

a) $\dfrac{2x^2}{(x^2+1)^2}$ L2

b) $\dfrac{4x^2-x}{(x^2+5)^2}$ L2

c) $\dfrac{x^2+1}{(x^2+2x+3)^2}$ L3

d) $\dfrac{-2x^2+2x-3}{(x^2+1)^2}$ L3

e) $\dfrac{x^3-x^2}{(x^2+3)^2}$ L3

Ans:

a) $\frac{2}{x^2+1}-\frac{2}{(x^2+1)^2}$

b) $\frac{-x-20}{(x^2+5)^2}+\frac{4}{x^2+5}$

c) $\frac{-2x-2}{(x^2+2x+3)^2}+\frac{1}{x^2+2x+3}$

LESSON ACTIVITY 8: [MATHEMATICAL INDUCTION]

SUB-STRAND	SLO NO.	SPECIFIC LEARNING OUTCOMES	SKILL LEVEL	SLO CODE	ACHIEVED
1.1	23	Prove a given mathematical statement is true by using the method of mathematical induction.	2	Cal1.1.2.7	
1.1	24	**prove** a given mathematical statement is true by using the method of mathematical induction, whereby the variable "n" in the statement to be proved is a power (the laws of indices are applied.	3	Cal1.1.3.3	

MATHEMATICAL INDUCTION

Induction is the process of generalizing from a repeated pattern from the past. It works on mathematical sentences involving a natural number n. we summarise these sentences in one statement, S(n).

The Two Induction Steps:

Any proof by induction usually requires two separate steps;

1. S(1) must be shown to be true. This gives a starting point for the chain.
2. Assuming that S(n) is true (induction hypothesis), we need to show that $S(n+1)$ must also be true.

If both (1) and (2) above hold, then the statement $S(n)$ must hold for any natural number n.

EXAMPLE A:

Use mathematical induction to prove that $1+2+3+...+n=\frac{n(n+1)}{2}$ for all positive integers n.

ANSWER:

- Let the statement S (n) :$1+2+3+...+n=\frac{n(n+1)}{2}$
- STEP 1: We first show that p (1) is true.

Left Side = 1 Right Side = $\frac{1(1+1)}{2}=1$

- Both sides of the statement are equal hence S (1) is true.
- STEP 2: We now assume that p (k) is true

$$1+2+3+...+k=\frac{k(k+1)}{2}$$

Show that S(k + 1) is true by adding (k + 1) to left hand side and replace k with $(k+1)$ on right hand side of the of the above statement .

LHS RHS

$$1+2+3+...+k+(k+1)=\frac{(k+1)((k+1)+1)}{2}$$ [by induction hypothesis]

LHS: $$=\frac{k(k+1)}{2}+(k+1)$$ [simplify]

$$=\frac{k(k+1)+2(k+1)}{2}$$

$$=\frac{(k+1)((k+2)}{2}$$

LHS = RHS

Hence $S(n+1)$ is true; so by the principle of mathematical induction, $S(n)$ is true for all n.

ACTIVITY 8A

Use mathematical induction to prove the following for all positive integers n.

1. $2+4+6+...+2n=n(n+1)$ L2
2. $1^2+2^2+3^2+...+n^2=\frac{n(n+1)(2n+1)}{6}$ L2
3. $1^3+2^3+3^3+...+n^3=\frac{n^2(n+1)^2}{4}$ L2

LESSON ACTIVITY 9: BINOMIAL THEOREM

SUB-STRAND	SLO NO.	SPECIFIC LEARNING OUTCOMES	SKILL LEVEL	SLO CODE	ACHIEVED
1.1	17	Expand and simplify expressions of the form $(x+y)^n$ for n = 3 or 4 using the Binomial Theorem.	1	Cal1.1.1.13	
1.1	18	Find specific terms and/or their coefficients in any expansion using Binomial Theorem, where n = 3 or 4.	2	Cal1.1.2.4	
1.1	19	Find the coefficients or constant term, or the term "independent of x" in expansions where n is greater than 4, using the Binomial theorem.	3	Cal1.1.3.2	

THE BINOMIAL THEOREM

The binomial theorem provides a method for expanding brackets without having to use repeated multiplication.

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + \dots + {}^nC_rx^{n-r}y^r + \dots + {}^nC_ny^n = \sum_{r=0}^{n} {}^nC_rx^{n-r}y^r$$

- The binomial coefficients, ${}^nC_r$, are the numbers in the relevant row of Pascal's Triangle
- The power of x decreases by one and the power of y increases by one as we move from left to right.
- The total power (power of x + power of y) is always equal to n for each term.

***Example A*:** Expand $(5+2y)^4$ using binomial theorem.

Answer: $(5+2y)^4 = {}^4C_0 5^4(2y)^0 + {}^4C_1 5^3(2y)^1 + {}^4C_2 5^2(2y)^2 + {}^4C_3 5^1(2y)^3 + {}^4C_4 5^0(2y)^4$

$= 1\times1\times625 + 4\times2y\times125 + 6\times4y^2\times25 + 4\times8y^3\times5 + 1\times16y^4\times1^4$

$= 625 + 1000y + 600y^2 + 160y^3 + 16y^4$

THE GENERAL TERM IN BINOMIAL EXPANSION

We define the **general term** in the expansion of $(p+q)^n$ to be:

$$T_{r+1} = {}^nC_rp^{n-r}q^r .$$

This enables us to find a specific term in an expansion without having to actually expand.

The term x^0 in the expansion of $(ax+b)^n$ is referred to as the **constant term**.

Example B:

Find the general term in the expansion of $(3x-4y)^{17}$

Answer: $$T_{r+1}={}^{n}C_{r}p^{n-r}q^{r}$$

Example C:

Determine the term of x^5 in the expansion of $(x-3)^8$

Answer:
$$T_{r+1}={}^{8}C_{3}(x)^{5}(-3)^{3}$$
$$T_{4}=56\times x^{5}\times(-27)$$
$$T_{4}=-1512x^{5}$$

Example D:

Find the constant term in the expansion of $\left(x^3-\frac{2}{x}\right)^8$

Answer: first we write down a formula for the general term:

$$T_{r+1}={}^{8}C_{r}(x^{3})^{8-r}(\frac{-2}{x})^{r}$$
$$T_{r+1}={}^{8}C_{r}x^{24-3r}(-2)^{r}x^{-r}$$
$$T_{r+1}={}^{8}C_{r}x^{24-4r}(-2)^{r}$$

The constant term is x^0 [the power of x must be equal to zero]

$$24-4r=0$$
$$r=6$$

So the constant term:

$$T_{7}={}^{8}C_{6}(-2)^{6}$$
$$T_{7}=28\times 64$$
$$T_{7}=1792$$

ACTIVITY 9A:

1. Use Binomial Theorem to expand and simplify:

   a) $(3x - 2y)^4$ L1

   b) $\left(y + \frac{1}{y}\right)^6$ L1

   c) $(1 + x)(1 - 3x)^6$ L2

2. Determine the third term in the expansion $(1 - 4x)^5$ L2

3. Find the constant term in the expansion of $\left(2x - \frac{1}{x^2}\right)^9$ L2

4. Find the coefficient of x^4 in $\left(2 + \frac{1}{x}\right)(2x - 3)^6$ L2

Ans:

1. a) $81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 - 16y^4$

LESSON ACTIVITY 10: SURDS

SUB-STRAND	SLO NO.	SPECIFIC LEARNING OUTCOMES	SKILL LEVEL	SLO CODE	ACHIEVED
1.1	25	Simplify sums, differences and products of surds.	1	Cal1.1.1.18	
1.1	26	Simplify quotients of surds (including rationalizing)	2	Cal1.1.2.6	
1.1	27	use and manipulate simple surds	1	Cal1.1.1.16	
1.1	28	Use and manipulate surds and other irrational numbers.	2	Cal1.1.2.9	
1.1	22	Solve straightforward surd equations and check solutions	2	Cal1.1.2.4	
1.2	15	solve surd equations, including simplifying surds, expanding surds and writing surds in simplest forms	2	Cal1.2.2.4	
1.2	16	solve surd equations, including rationalizing the denominator	3	Cal1.2.3.6	

RATIONAL AND IRRATIONAL NUMBERS

The set of rational numbers $\mathbb{Q}$ can be defined as $\left\{x : x = \frac{a}{b} ; a \in I ; b \in \mathbb{N}\right\}$, $\mathbb{Q}$ includes all fractions so that any division has an answer.
The set of all real numbers $\mathbb{R}$, is all the numbers on the number lines including irrational numbers such as roots and π which cannot be written as fractions.

SIMPLIFYING SURDS

Some of these surds can be simplified if the number under the root sign is a multiple of a perfect square. Recall that perfect squares are numbers such as 4, 9, 16, 25, 36, 49, etc. When simplifying surds, the aim is to eventually write the surd with the lowest possible whole number under the surd sign. The method involves writing the number under the surd as a product of a perfect square and another number.

Example A: Simplify $\sqrt{72}$

Answer: $\sqrt{72} = \sqrt{36 \times 2} = \sqrt{36} \times \sqrt{2} = 6\sqrt{2}$ (writing 72 as a multiple of 36, the largest factor that is a perfect square)

Example B: Simplify $\sqrt{25a^3b^2c^7}$

Answer: $\sqrt{25a^3b^2c^7} = \sqrt{25a^2b^2c^6 \times ac}$

$= 5abc^3\sqrt{ac}$

ACTIVITY 10A

1. Simplify the following surds expression:

a) $\sqrt{80}$ L1

b) $\sqrt{300}$ L1

c) $2\sqrt{12}$ L1

d) $4\sqrt{125}$ L1

SUMS AND DIFFERENCES OF SURDS

Surds with the same number under the root sign behave in the same way as *like terms* in algebra. They can be added and subtracted.

Example C: Simplify $\sqrt{50} - \sqrt{8}$

Answer: $\sqrt{50} - \sqrt{8} = 5\sqrt{2} - 2\sqrt{5}$

$= 3\sqrt{2}$

Example D: Simplify $\sqrt{18} + \sqrt{20} + \sqrt{45}$

Answer: $\sqrt{18} + \sqrt{20} + \sqrt{45} = 3\sqrt{2} + 2\sqrt{5} + 3\sqrt{5} \quad = 3\sqrt{2} + 5\sqrt{5}$

No further simplification is possible.

ACTIVITY 10B:

1. Simplify these surds expressions:

a) $6\sqrt{3} + 2\sqrt{5} - 2\sqrt{3} + 4\sqrt{5}$ L1

b) $\sqrt{18} + \sqrt{50} - \sqrt{32}$ L1

c) $\sqrt{98} - \sqrt{192} + \sqrt{50}$ L1

MULTIPLYING SURDS

When multiplying surds, simply multiply the numbers under the root signs.
Two rules that are useful when working with surds are:

- $\sqrt{pq} = \sqrt{p} \times \sqrt{q}$
- $\sqrt{\frac{p}{q}} = \frac{\sqrt{p}}{\sqrt{q}}$

Example E: Simplify $2\sqrt{3} \times 5\sqrt{48}$

Answer: $2\sqrt{3} \times 5\sqrt{48} = 2 \times 5 \times \sqrt{3} \times 4\sqrt{3}$

$$= 40 \times \sqrt{3 \times 3}$$

$$= 40 \times 3$$

$$= 120$$

Note: Expressions where there are surd terms inside several pairs of brackets can be expanded in the same way as algebraic expressions.

Example F: Simplify $\left(2\sqrt{3} + \sqrt{15}\right)\left(\sqrt{5} - \sqrt{27}\right)$

Answer: $\left(2\sqrt{3} + \sqrt{15}\right)\left(\sqrt{5} - \sqrt{27}\right) = \left(2\sqrt{3} + \sqrt{15}\right)\left(\sqrt{5} - 3\sqrt{3}\right)$ [simplify surds first]

$$= 2\sqrt{3 \times 5} - 6\sqrt{3 \times 3} + \sqrt{15 \times 5} - 3\sqrt{15 \times 3}$$

$$= 2\sqrt{15} - 18 + \sqrt{5^2 \times 3} - 3\sqrt{3^2 \times 5}$$

$$= 2\sqrt{15} - 18 + 5\sqrt{3} - 9\sqrt{5}$$

ACTIVITY 10C:

1. Multiply these surds expressions and simplify:

   a) $4\sqrt{6} \times 3\sqrt{2}$ L1

   b) $\sqrt{3}\left(\sqrt{5} - \sqrt{3}\right)$ L1

2. Expand and simplify these expressions:

   a) $\left(2\sqrt{2} + \sqrt{3}\right) + \left(3\sqrt{2} - 1\right)$ L2

   b) $\left(3\sqrt{5} + 2\right)^2$ L2

RATIONALISING SURDS

When dealing with surd expressions, the convention is that we give answers with whole numbers in the denominator (bottom line).
This process is called **rationalising the denominator**.

Example G: Write $\frac{2}{\sqrt{5}}$ as a fraction with a rational denominator.

Answer: Multiply top and bottom by $\sqrt{5}$. This is essentially the same as multiplying by 1, which does not change the value of the fraction. We multiply by $\sqrt{5}$ because $\sqrt{5} \times \sqrt{5}$ gives 5, which removes the surd form in the denominator, thus:

$$\frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

$$= \frac{2\sqrt{5}}{5}$$

CONJUGATE SURDS

The type of problem above – where there is a surd term only in the denominator – can be extended to ones where there is a mixture of a number and a surd in the denominator. We still follow the convention of simplifying so that any surd terms are removed from the denominator of fractions. This process involves working with **conjugate** surds.

If $\boldsymbol{a} + \sqrt{\boldsymbol{b}}$ is a surd then its conjugate surd is $\boldsymbol{a} - \sqrt{\boldsymbol{b}}$.
Similarly, if $\boldsymbol{a} - \sqrt{\boldsymbol{b}}$ is a surd then its conjugate is $\boldsymbol{a} + \sqrt{\boldsymbol{b}}$

In examples where fractions have to be simplified and there are surds expressions in the *denominator*, the first major step is to multiply top and bottom by the conjugate of the denominator.

Example G: Simplify $\frac{2}{1-\sqrt{5}}$

Answer: Conjugate of $1-\sqrt{5}$ is $1+\sqrt{5}$ so multiply top and bottom by the same.

$$\frac{2}{1-\sqrt{5}} = \frac{2}{1-\sqrt{5}} \times \frac{1+\sqrt{5}}{1+\sqrt{5}}$$

$$= \frac{2+2\sqrt{5}}{1-\sqrt{5}+\sqrt{5}-\sqrt{25}}$$

$$= \frac{2+2\sqrt{5}}{1-5}$$

$$= \frac{2(1+\sqrt{5})}{-4}$$

$$= \frac{-1-\sqrt{5}}{2}$$

ACTIVITY 10D:

1. Write these expression with rational denominators:

a) $\frac{1}{\sqrt{3}}$ L1

b) $\frac{4}{9\sqrt{3}}$ L1

2. Write $\frac{2}{3+\sqrt{7}}$ with a rational denominator. L2

3. Rewrite $\frac{2-3\sqrt{3}}{4+\sqrt{3}}$ in the form $a + b\sqrt{3}$ where a and b are rational numbers. L2

4. Rationalize the denominator: L2

(a) $\frac{2}{5\sqrt{3}}$ (b) $\frac{2}{4-\sqrt{3}}$ (c) $\frac{3\sqrt{3}}{1+\sqrt{3}}$ (d) $\frac{1-\sqrt{2}}{1+\sqrt{2}}$ (e) $\frac{2}{4-\sqrt{3}} + \frac{2}{\sqrt{3}}$

SOLVING EQUATIONS INVOLVING SURDS

To solve equations involving surds (irrational numbers expressed as square roots, cube roots, etc.), these expression must be raised to appropriate (inverse) power. Always check solutions after squaring so that any 'false solutions' can be rejected (substitute answers into the original equation)

Example A: Solve $3x - 2 = \sqrt{x+10} + 2x$

Answer: $3x - 2 = \sqrt{x+10} + 2x$

$x - 2 = \sqrt{x+10}$

$(x * 2)^2 = x + 10$

$x^2 - 4x + 4 = x + 10$

$x^2 - 5x - 6 = 0$

$(x-6)(x+1) = 0$

$x = 6 \; or \; x = \; -1$

Check solutions: $x = 6, \;\; 18 - 2 = \sqrt{16} + 12$ [equal, correct]

$x = -1, -3 - 2 = \sqrt{9} - 2$ [not equal, incorrect]

Therefore the only solution is $x = 6$

ACTIVITY 10E

1. Solve the following equations:
   a) $\sqrt{x+5} = 2x - 3$ L2

   b) $3\sqrt{x-2} = x - 2$ L2

   c) $\sqrt{x+3} = 2x$ L2

   d) $\frac{x(2-x)}{\sqrt{1-x^2}} - \sqrt{1-x^2} = 1$ L2

2. Solve the following equations for x in terms of p:

   a) $\sqrt{x+p} = \sqrt{x} + 3$ L2

   (c) $p(x - 2) = 3\sqrt{x}$ L2

   b) $2\sqrt{x+1} - p\sqrt{x} = 0$ L2

3. Solve the equation for x in terms of k: L3
   $\sqrt{\frac{x}{1-x}} + \sqrt{\frac{1-x}{x}} = \frac{k}{6}$

SUB-STRAND 1.3 COMPLEX NUMBERS

LESSON ACTIVITY 1:

SUB-STRAND	SLO NO.	SPECIFIC LEARNING OUTCOMES	SKILL LEVEL	SLO CODE	ACHIEVED
1.3	1	Simplify sums, differences and products of complex numbers expressed in rectangular form.	1	Cal1.3.1.1	
1.3	2	Simplify quotients of complex numbers expressed in rectangular form.	2	Cal1.3.2.1	

COMPLEX NUMBERS

There are no real solutions to some polynomial equations, $x^2 + 1 = 0$ (since all real numbers $x, x^2 \geq 0$ therfore $x^2 + 1 \geq 1$) for this reason, a new number is defined:

$$\boxed{\boldsymbol{i = \sqrt{-1}}} \quad \text{Or} \quad \boxed{\boldsymbol{i^2 = -1}}$$

We set up a system of new numbers that enable us to 'solve' all algebraic equations, especially those quadratic equations with a discriminant $(b^2 - 4ac)$ less than 0.

We represent the set of complex numbers by the letter $\mathbb{C}$
The letter z is used to represent an arbitrary complex number.
Any complex number, z, can be written as:

$$\boldsymbol{z = x + iy}$$

- We call x the **real** part of z, and write $x = \text{Re}(z)$.
- We call y the **imaginary** part of z, and write $y = \text{Im}(z)$.

ADDITION AND SUBTRACTION OF COMPLEX NUMBERS

To add (or subtract) complex numbers, add (or subtract) real parts and add (or subtract) imaginary parts.

Example A: if $u = 2 + 3i$ and $v = 1 - 4i$,find:

a) $u + v$
b) $u - v$

Answer: a)
$$\begin{aligned} u + v &= (2 + 3i) + (1 - 4i) \\ &= (2 + 1) + (3i - 4i) \\ &= 3 - i \end{aligned}$$

c)
$$\begin{aligned} u - v &= (2 + 3i) - (1 - 4i) \\ &= (2 - 1) + (3i + 4i) \end{aligned}$$

$$= 3 + 7i$$

MULTIPLYING COMPLEX NUMBERS

Multiplying complex numbers, expand brackets in usual way, remembering that $i^2 = -1$

EXAMPLE B: Expand and simplify $(2 + 3i) \times (5 - 3i)$

Answer: $(2 + 3i) \times (5 - 3i) = 2(5 - 3i) + 3i(5 - 3i)$

$$= 10 + 15i - 6i - 9i^2$$

$$= 10 + 9i - 9 \times -1$$

$$= 10 + 9i + 9$$

$$= 19 + 9i$$

- **Powers of *i***

Any complex number, no matter how many times it multiplied by itself, always gives another complex number. This is because any power of i simplifies to $1, -1, i, or - i$. Powers of i can be simplified by treating any power of i that is a multiple of 4 as having the value 1.

Example C: Simplify $2i^6 \times 3i^5$

Answer: $2i^6 \times 3i^5 = 6i^{11}$

$$= 6i^3 \times i^8$$

$$= 6i^2 \times i \times (1) \quad [i^2 = -1]$$

$$= 6 \times -1 \times i$$

$$= -6i$$

- **Complex Conjugates**

The conjugates of complex number is used in division. If z is a complex number $x + iy$ then its conjugate i is defined by:

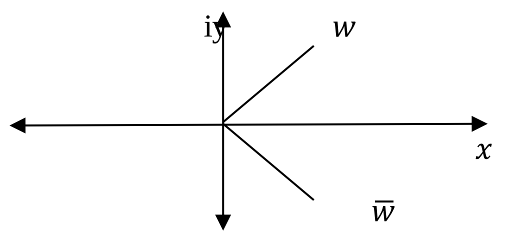
$$\bar{\boldsymbol{z}} = \boldsymbol{x} - \boldsymbol{iy}$$

The product of a complex number and its conjugate is always real.

Example D: Evaluate the conjugate of $w = 1 + 3i$

Answer: $\bar{w} = 1 - 3i$

The geometric relationship between w and $\bar{w}$ can be found using the Argand diagram.



It is clear that $\bar{w}$ is the image of w under the reflection in the real axis.

DIVISION OF COMPLEX NUMBERS

When one complex number is divided by another, the result is also a complex number. To simplify a quotient of complex numbers, multiply the numerator and denominator by its conjugate of the denominator. This creates an expression with a denominator which is a real number.

Example E: If $z = 17 + 19i$ and $w = 2 + 3i$, evaluate $\frac{z}{w}$ and express answer in the form $a + ib$

Answer: $\frac{17+19i}{2+3i} = \frac{17+19i}{2+3i} \times \frac{2-3i}{2-3i}$

$$= \frac{34+38i-51i-57i^2}{4+6i-6i-9i^2}$$

$$= \frac{34-13i-57\times(-1)}{4-9\times(-1)}$$

$$= \frac{34+57-13i}{4+9}$$

$$= \frac{91-13i}{13}$$

$$= 7 - i$$

ACTIVITY 1A:

1. Simplify the following:

   a) i^{30} L1
   b) $2i^9 \times 3i^{12}$ L1
   c) $(3 - i)(1 + i)^2$ L2
   d) $\frac{(1-2i)(1-i)(1+i)}{(3-2i)^2}$ L3

2. The complex numbers u, v and w are as follows:

   $\boldsymbol{u = 3 + 4i}$ $\boldsymbol{v = -2 + i}$ $\boldsymbol{w = 5 - i}$

   Calculate the following:

   a) $2u + 3v$ L1

   b) $\bar{u} - 4v + 2w$ L2

   c) $v\bar{w}$ L2

   d) $u^2 - 6v^2$ L2

   e) $\frac{v}{w}$ L2

f) $\frac{2w}{3\bar{v}}$ L2

3. Express $\frac{1}{2i} - \frac{1}{1-2i}$ in the form $a + ib$ L2

Ans:

1.a) -1; b) 6i; c) 2+6i,

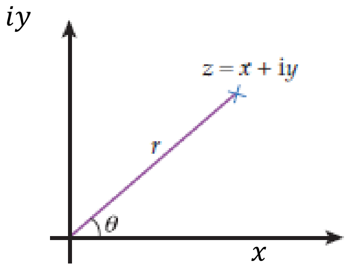
2.a)11i; b)21-10i

LESSON ACTIVITY 2: COMPLEX NUMBERS, ARGAND DIAGRAM

SUB-STRAND	SLO NO.	SPECIFIC LEARNING OUTCOMES	SKILL LEVEL	SLO CODE	ACHIEVED
1.3	6	**Interpret, manipulate and use** graphical representations of complex numbers, using polar and rectangular form on an Argand diagram. These problems include a point on the complex plane can be written in rectangular or polar form.	2	Cal1.3.2.2	
1.3	7	**Interpret, manipulate and use** graphical representations of complex numbers, using polar and rectangular form on an Argand diagram. These problem include regions that are shaded on a line ($a \leq Re/Im(Z) \leq b$) that can be linked back to its rectangular or polar form.	3	Cal1.3.3.1	
1.3	8	**Interpret manipulate and use** graphical representations of complex numbers, using polar and rectangular form on an Argand diagram. (These problems include a region involving a circle that can be linked back to its rectangular/polar form. E.g. $-2 \leq \|z\| < 1$)	4	Cal1.3.4.1	

THE ARGAND DIAGRAM

The complex number $z = x + iy$ can be plotted as ordered pair on a complex plane known as Argand Diagram. The axes on the Agrand diagram are labelled x and y or Re (real) and Im (imaginary)



GEOMETRICAL PROPERTIES OF COMPLEX NUMBER.

The complex numbers can be graphed, their geometrical properties can be considered.

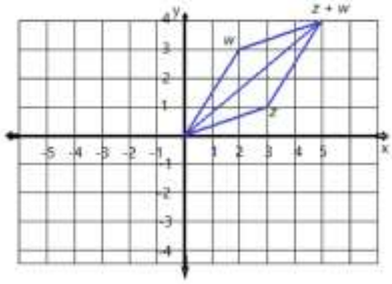
Addition of Complex Numbers.

The parallelogram law for adding vectors is applied to the addition of z and w are shown below.

Note: vectors are drawn without arrows in the following representations.

Example: Find the result of adding $z = 3 + i$ and $w = 2 + 3i$ on the Argand diagram.

Answer: $$z + w = 5 + 4i$$

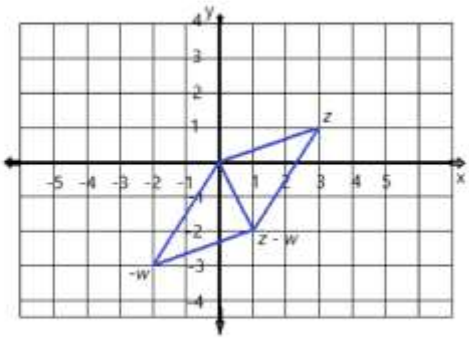


Subtraction of Complex Number.

Subtraction is regarded as the addition of negative value. Thus $z - w = z + -w$

Example: Find the result of subtracting $z = 3 + i$ and $w = 2 + 3i$ on the Argand Digram.

Answer: $$z - w = 1 - 2i$$



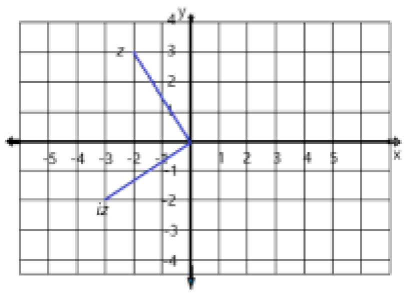
Multiplication of z by *i*

The example below illustrates that multiplication by i rotates z about 0 through and angle of 90°, anticlockwise.

Example: Suppose $z = -2 + 3i$ and is multiplied

Answer: $i(-2 + 3i) = -2i + 3i^2$

$= -3 - 2i$



ACTIVITY 2A:

1. Using the parallelogram law of addition. Represent the addition and subtraction of these complex numbers on the Argand diagram.

a) $(1+2i)+(-3+i)$ L2

b) $2i+(4-3i)$ L2

c) $(1+3i)-(2-i)$ L2

d) $(-2+i)-(3+i)$ L2

2. Draw the position of these complex numbers on the Argand diagram:

a) $3cis\left(\frac{2\pi}{3}\right)$ L2

b) $cis\left(-\frac{\pi}{6}\right)$ L2

c) $\sqrt{2}cis\left(-\frac{\pi}{2}\right)$ L2

3 If $=-1+2i$, use the Argand diagram to explain the geometrical relationship between:

a) z and i^2z L3 b) z and i^3z L3 c) z and $\bar{z}$ L4

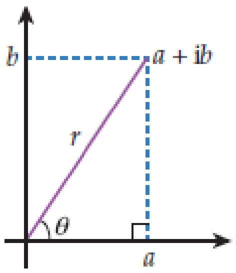
LESSON ACTIVITY 3:

SUB-STRAND	SLO NO.	SPECIFIC LEARNING OUTCOMES	SKILL LEVEL	SLO CODE	ACHIEVED
1.3	3	Convert between rectangular $(a + ib)$ and polar $(rcis\theta)$ form.	1	Cal1.3.1.2	
1.3	4	use Argand diagrams to represent complex number in the forms a + ib , rcis θ The Argand diagram is either represented as an intercept on the x-axis or y-axis.	1	Cal1.3.1.3	
1.3	29	use Argand diagrams to represent complex number in the forms a + ib , rcis θ The Argand diagram is plotted as a point on any of the four quadrants. Example: It has BOTH a Real and Imaginary component, i.e. $Z = -2 + 3i$	2	Cal1.3.2.2	

MODULUS OF A COMPLEX NUMBER.

The distance of $z = a + ib$ from the origin is called the modulus of z, written as |z| or 'r'

The modulus of $z = a + ib$ is defined by: $|z| = r = \sqrt{x^2 + y^2}$



Example 1:

Find the modulus of the complex number $z = -3 + 4i$

Answer: $r = \sqrt{(-3)^2 + 4^2}$

$r = 5$

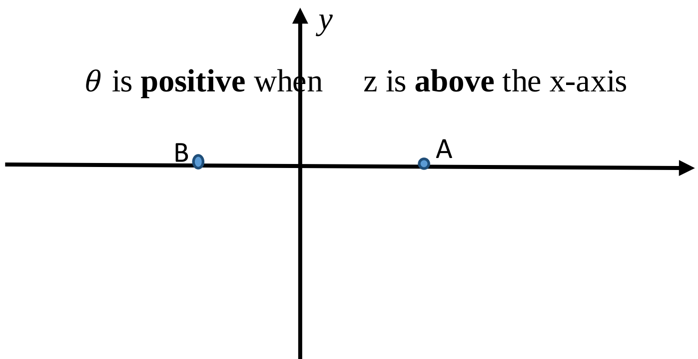
ARGUMENT OF COMPLEX NUMBER

The Argument of **z** written as (arg(**z**)) is angle θ that r makes with positive $x - axis$ where

$-\pi \leq \theta \leq \pi$

$$\arg(z) = \theta, \quad where\ tan\theta = \frac{y}{x}$$

The convention are summarised in the following diagram.



x

When z is at the **point B, $\boldsymbol{\theta = \pi}$** when z is at the **point A, $\boldsymbol{\theta =}$ 0**

θ is **negative** when z is **below** the x-axis

Example: If $z = 3 - 4i$, plot z on Argand the diagram and determine its modulus and argument.

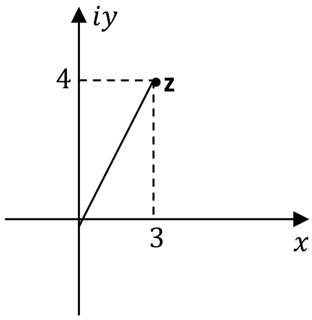
Answer: z is plotted as point (3,-4) as shown on the diagram.

$$|z| = \sqrt{3^2 + (-4)^2}$$
$$= \sqrt{25}$$
$$= \mathbf{5}$$

$$\text{Arg}(z) = \theta, \text{ where } \quad tan\theta = \frac{-4}{3}$$

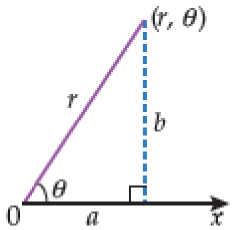
$$\theta = tan^{-1}(\frac{-4}{3})$$

$$\theta = -53.1°$$



WRITING COMPLEX NUMBERS IN POLAR FORM

The complex number has been written in the form $a + ib$, which is called **rectangular form** or **Cartesian form**. When a complex number is written in terms of its **modulus** and **argument** it is said to be in **Polar form**.



Using trigonometry in the figure:

$$\frac{a}{r} = \cos(\theta) \text{ and } \frac{b}{r} = \sin(\theta)$$

$$\therefore a = r\ cos(\theta)\ \ and\ b = r\ sin(\theta)$$

$$\boldsymbol{a + ib = rcos(\theta) + rsin(\theta)}$$
$$\boldsymbol{= r[\cos(\theta) + rsin(\theta)]}$$
$$\boldsymbol{= rcis\ (\theta)}$$

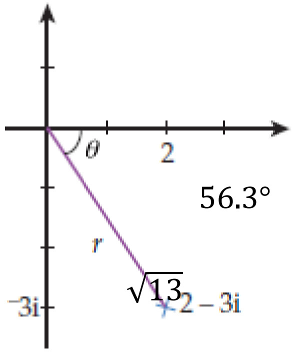
Thus in polar form:

$$\boldsymbol{z = rcis(\theta)} \quad \text{where} -\pi < \theta < \pi$$

❖ **Convert Rectangular form to Polar Form**

It is usually helpful to draw a diagram when converting complex numbers from one form to another.

Example A: Convert the complex number $\boldsymbol{z = 2 - 3i}$ into polar form and draw the Argand diagram.



Answer: Modulus $(r) = \sqrt{2^2 + (-3)^2}$

$r = \sqrt{13}$

Argument $(\theta) = tan\theta = \frac{-3}{2}$

$\theta = tan^{-1}(\frac{-3}{2})$

$\theta = -56.3°$

$\therefore \boldsymbol{z} = \sqrt{13}\ cis\ (-56.3°)$

Use calculator:

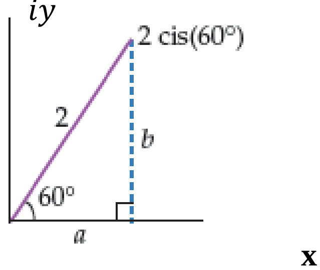
r: Pol (→ 2 → , → 3) = 3.61

θ: RCL → F = -56°

∴ **2 – 3*i* = 3.606 *cis* (-56.3°)**

❖ **Convert Polar form to Rectangular Form**

Example 1: Convert $\boldsymbol{z} = 2cis(60°)$ into rectangular form



Answer:

$z = 2[\cos(60°) + \sin(60°)]$
$z = 2[0.5 + 0.8660i]$

$\boldsymbol{z = 1 + 1.732i}$

Use Calculator:

Real Part: Shift → Rec (→ 2 → , → 60) = 1 (*Do note erase)*

Im Part: RCL → F = 1.73

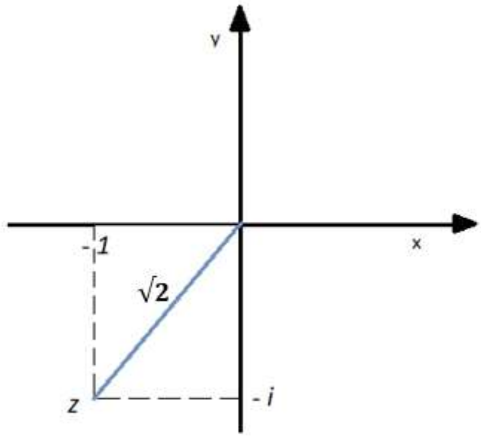
∴ ***z*** *= 2cis(60°) = 1 + 1.73i*

Example 2: Convert $z = \sqrt{2}cis\left(\frac{-3\pi}{4}\right)$ to Cartesian form and draw the Argand diagram.

Answer: $z = \sqrt{2}\left[\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right]$

$z = \sqrt{2}\left[\left(\frac{-1}{\sqrt{2}}\right) + \left(\frac{-1}{\sqrt{2}}\ i\right)\right]$ *{using exact values}*

$z = -1 - i$



ACTIVITY 3A:

1. Convert the following complex numbers to polar form and then draw them on the Argand diagram showing the argument and modulus for each.

   a) $w = 5 + 2i$ L1

   b) $z = -2 + 3i$ L1

   c) $u = -4 - i$ L1

   d) $v = 3 - 3i$ L1

   e) $z = \frac{-1+\sqrt{3}i}{2}$ L1

2. Convert the following complex numbers to rectangular form and then draw them on the Argand diagram.

   a) $w = 2cis\left(\frac{-5\pi}{4}\right)$ L2

   b) $v = 3cis(-130°)$ L2

   c) $u = 3cis\left(\frac{\pi}{6}\right)$ L2

   d) $z = \sqrt{5}cis(45°)$ L2

MULTIPLICATION AND DIVISION OF A COMPLEX NUMBER IN POLAR FORM.

It can be proved that if $z_1 = r_1 cis\theta_1,\quad z_2 = r_2 cis\theta_2$ then

- **Multiplication:**

$z_1z_2 = r_1cis\theta_1 \times r_2cis\theta_2 = r_1r_2cis(\theta_1 + \theta_2)$ ***Multiply** moduli and **add** argument*

Division: $\boxed{\frac{z_1}{z_2} = \frac{r_1cis\theta_1}{r_2cis\theta_2} = \frac{r_1}{r_2}cis(\theta_1 - \theta_2)}$

***Divide** moduli and **subtract** argument*

Examples: If $u = 2cis\left(\frac{\pi}{4}\right)$ and $v = 3cis\left(\frac{\pi}{2}\right)$, then give uv and $\frac{u}{v}$ in polar form.

Answer:

$$\boldsymbol{uv} = (2 \times 3)cis\left(\frac{\pi}{4} + \frac{\pi}{2}\right)$$
$$= 6cis\left(\frac{3\pi}{4}\right)$$

$$\frac{\boldsymbol{u}}{\boldsymbol{v}} = \frac{2}{3}cis\left(\frac{\pi}{4} - \frac{\pi}{2}\right)$$
$$= \frac{2}{3}cis\left(-\frac{\pi}{4}\right)$$

Example: Divide $\frac{4cis(120°)}{2cis(30°)}$ and express answer in the rectangular ($+ib$) form.

Answer:

$$\frac{4cis(120°)}{2cis(30°)} = \frac{4}{2}cis(120° - 30°)$$
$$= 2cis\,(90°)$$
$$= 2[\cos(90°) + isin(90°)]$$
$$= 2[(0 + i)]$$
$$= 2i$$

DE MOIVRE'S THEOREM.

Using the approach we learnt in the previous section for the multiplication of two complex numbers in the polar form.

If $\boldsymbol{z = rcis\theta}$ then $z^2 = (rcis\theta)^2$
$$= rcis\theta \times rcis\theta$$
$$= r^2cis(2\theta)$$

Similarly $z^3 = (rcis)^3$
$$= rcis\theta \times rcis\theta \times rcis\theta$$
$$= r^2cis(2\theta) \times rcis\theta$$
$$= r^3cis(3\theta)$$

In general terms, If $\boldsymbol{z = rcis\,\theta}$ then **De Moivre's Theorem** states:

$$\boxed{\begin{array}{l}(rcis\theta)^n = r^ncisn\theta\,, \\ \quad \boldsymbol{z^n = r^ncis(n\theta)}\end{array}}$$

where n is an integer and argument must lie within range $-\boldsymbol{\pi} < \theta \leq \pi$

Example: Simplify z^4 if $z = 2cis\left(\frac{\pi}{2}\right)$

Answer:
$$z^4 = \left(2cis\left(\frac{\pi}{2}\right)\right)^4$$
$$= 2^4\left(cis\left(\frac{\pi}{2}\right)\right)^4$$
$$= 16cis\left(\frac{\pi}{2} \times 4\right) \quad \textit{[using De Moivre's theorem]}$$
$$= 16cis(2\pi) \quad \textit{[subtract } 2\pi \textit{ from the argument]}$$
$$= 16cis(0) \quad \textit{[polar form]}$$
$$= 16[\cos(0) + isin(0)]$$
$$= 16[(1 + 0i)]$$
$$= 16 \quad \textit{[rectangular form]}$$

Example: if $z = \left(cos\left(\frac{\pi}{3}\right) - isin\left(\frac{\pi}{3}\right)\right)$, represent z^2 on the Argand diagram.

Answer: $z = \left(cos\left(\frac{\pi}{3}\right) - isin\left(\frac{\pi}{3}\right)\right)$

$= \left(cos\left(-\frac{\pi}{3}\right) + isin\left(-\frac{\pi}{3}\right)\right)$, *[since $cos(-\theta) = cos(\theta)$ and $sin(-\theta) = -sin\theta$ and $r = 1$]*

$$z^2 = \left(cos\left(-\frac{\pi}{3}\right) + isin\left(-\frac{\pi}{3}\right)\right)^2 \quad \textit{[squaring both sides]}$$
$$= \left(cos\left(-\frac{2\pi}{3}\right) + isin\left(-\frac{2\pi}{3}\right)\right) \quad \textit{[applying De Moivre's theorem]}$$
$$= cis\left(-\frac{2\pi}{3}\right) \quad \textit{[modulus}(r) = 1 \textit{ and argument}(\theta) = \frac{-2\pi}{3} \textit{ or } -120°\textit{]}$$

z^2 is represented on the Argand diagram Fig 1 below:

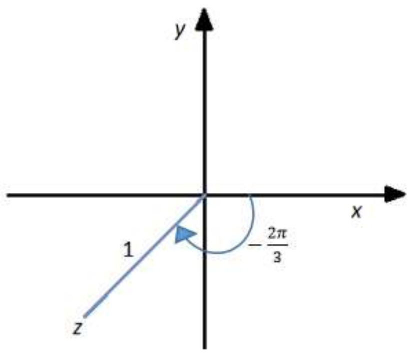


Fig 1

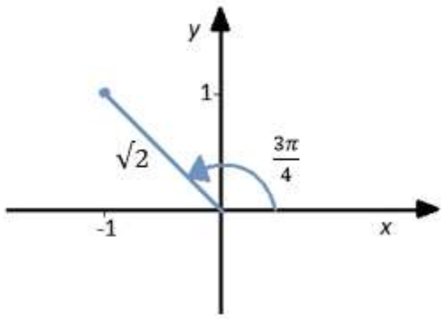


Fig 2

Example: Evaluate $(-1 + i)^{12}$

Answer: Firstly write the complex number in polar form:

Let $z = -1 + i$ then modulus $|z| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

Argument: $tan\theta = \frac{1}{-1}$ then $\theta = tan^{-1}(-1) = -\frac{\pi}{4}$

[since $(-1+i)$ is in Quadrant 2, add π to θ, see diagram Fig 2 above]

$\therefore\ \theta = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$

Hence $z = \sqrt{2}cis\left(\frac{3\pi}{4}\right)$ *[expressing z in polar form]*

$$z^{12} = \left(\sqrt{2}cis\left(\frac{3\pi}{4}\right)\right)^{12}$$

$= \left(\sqrt{2}\right)^{12} cis\left(\frac{3\pi}{4} \times 12\right)$ *[apply De Moivre's theorem]*

$= 2^6 cis(9\pi)$ *[subtract 8π from the argument]*

$= 64\ cis\ (\pi)$

$= 64[\cos(\pi) + i\sin(\pi)]$

$= 64(-1 + 0i)$

$= -64$

ACTIVITY 3B

1. u and v are two complex numbers where $u = 2cis\left(\frac{\pi}{4}\right)$ and $v = 6cis\left(\frac{3\pi}{7}\right)$

   a) Find uv expressing your answer in polar form, $rcis\theta$ L1

   b) Find $u \div v$, expressing your answer in rectangular form, $a + ib$ L2

2. If $z = 3cis(-120°)$ and $w = 2cis(135°)$

   a) Plot z and w on an Argand diagram L2

   b) Find zw in polar form L1

   c) Find $\frac{w^2}{z}$ in polar form. L2

3. Using De Moivre's theorem to simplify the following and then convert to rectangular form.

   a) $(2cis(60°))^3$ L1

   b) $(5cis(120°))^4$ L1

   c) $\left(\sqrt{3}cis(\frac{\pi}{3})\right)^3$ L1

   d) $\left(2cis(-\frac{2\pi}{3})\right)^6$ L1

4. Using De Moivre's theorem to simplify the following and then give your in rectangular form.

   a) $(3 - 2i)^5$ L3

b) $\left(-\sqrt{2}+\sqrt{2}i\right)^3$ L3

5. By using De Moivre's Theorem show that $\frac{\left(\sqrt{3}-i\right)^9}{(1+i)^7}$ can be simplified to $-32(1-i)$. L3

<u>**LESSON ACTIVITY 4: ROOTS OF COMPLEX NUMBERS**</u>

SUB-STRAND	SLO NO.	SPECIFIC LEARNING OUTCOMES	SKILL LEVEL	SLO CODE	ACHIEVED
1.3	9	**find** roots over the complex number system for polynomial equations with real coefficients, including the special case of the nth roots of *a* that come from solving equations of the form $z^n = a$, making links with their graphs. (The equation of the complex number has only a Real component or Imaginary component, but not both. Example: Z = 2 or Z = -i)	2	Cal1.3.2.4	
1.3	10	**find** roots over the complex number system for polynomial equations with real coefficients, including the special case of the nth roots of *a* that come from solving equations of the form $z^n = a$, making links with their graphs. (The equation of the complex number has both Real and Imaginary components and both components are whole numbers. Example: Z = -3 + 2i)	3	Cal1.3.3.2	
1.3	11	**find** roots over the complex number system for polynomial equations with real coefficients, including the special case of the nth roots of *a* that come from solving equations of the form $z^n = a$, making links with their graphs. (The equation of the complex number has both Real and Imaginary components. One of the components must include a rational/surd. Example: Z = $\sqrt{2} + 2i$)	3	Cal1.3.3.3	

POLYNOMIAL WITH COMPLEX FACTORS

Here are three different types of polynomials with examples of each.

Types of polynomials	Example
1. A polynomial over R (has real coefficients) and factored over R (with real factors)	$x^2+5x+6=(x+2)(x+3)$
2. A polynomial over R (has real coefficients) and factored over C (with complex factors)	$x^2+2x+2=(x+1+i)(x+1-i)$
3. A polynomial over C (has complex coefficients) and factored over C (with complex factors)	$x^2+5i-6=(x+2i)(x+3i)$

FUNDAMENTAL THEOREM OF ALGEBRA

The **fundamental theorem of algebra** states: any polynomial of degree *n* has exactly *n* (possible repeated) roots over C (and hence *n* possible repeated factors).

COMPLEX NUMBERS AND QUADRATICS

SUM OF TWO SQUARES

A new result of the **sum of two squares**. By simple multiplication, it is easy to see that:

$$a^2 + b^2 = (a + ib)(a - ib)$$

Notice that this is the same as the factorisation of the difference of two squares, but with *i* inserted in each factor.

Example: Factorise both of these expressions to linear factor.

a) $5x^2 + 20$

b) $16x^4 - 625$

Answer: use the 'sum of two squares'

a) $5x^2 + 20 = 5(x^2 + 4) = 5(x + 2i)(x - 2i)$

b) $16x^4 - 625 = (4x^2 - 25)(4x^2 + 25)$

$= (2x + 5)(2x - 5)(2x + 2i)(2x - 2i)$

COMPLETING THE SQUARE WHEN THERE ARE COMPLEX ROOTS

- Follow the procedures for completing the squares
- Insert $\boldsymbol{i^2}$ when there is a negative number under the square root.

Example*:* Solve the equation $x^2 + 4x + 13 = 0$ and hence find the factors using completing the square method.

Answer:

$$x^2 + 4x + 13 = 0$$
$$x^2 + 4x = -13$$
$$x^2 + 4x + 4 = -13 + 4$$
$$(x + 2)(x + 2) = -9$$
$$(x + 2)^2 = -9$$
$$(x + 2)^2 = 9i^2 \quad [\text{insert } i^2 = -1]$$
$$x + 2 = \pm 3i$$

$$x = -2 \pm 3i$$

There are two solutions $x = -2 + 3i$ and $x = -2 - 3i$

We can use the solutions to write the complete factorisation:

$$x^2 + 4x + 13 = [x - (-2 + 3i)][x - (-2 - 3i)]$$
$$= (x + 2 - 3i)(x + 2 + 3i)$$

THE QUADRATIC FORMULA WHEN THERE ARE COMPLEX ROOTS

- For quadratic equation, use the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Insert i^2 when the discriminant is negative.

Example: Solve $x^2 - 2x + 4 = 0$, and hence factorise $x^2 - 2x + 4$

Answer: $a = 1. b = -2, c = 4$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{2 \pm \sqrt{-12}}{2}$$

$$x = \frac{2 \pm 2\sqrt{3i^2}}{2}$$

$$x = \frac{2(1 \pm \sqrt{3}i)}{2}$$

$$x = 1 \pm \sqrt{3}i$$

The two roots are $\boldsymbol{x = 1 + \sqrt{3}i}$ and $\boldsymbol{x = 1 - \sqrt{3}i}$

The factors are given by: $[x - (1 + \sqrt{3}i)[x - (1 - \sqrt{3}i)]$ or $(x - 1 - \sqrt{3}i)(x - 1 + \sqrt{3}i)$

WRITING DOWN A POLYNOMIAL GIVEN ITS ROOTS

When we know the roots of a polynomial, we can write down its factors and then expand these to obtain the polynomial. Sometimes the polynomial has **repeated** roots. If a root is repeated twice we refer to it as having multiplicity 2; if it is repeated three times then it has multiplicity 3, and so on.

Examples 1: Give the polynomial with roots $-i$ and $2i$.

Answer: $(x + i)(x - 2i) = x(x - 2i) + i(x - 2i)$

$$= x^2 - 2ix + ix - 2i^2$$
$$= x^2 - ix + 2$$

Examples 2: Give the polynomial with roots 1 and i (multiplicity 2)

Answer: $(x-1)(x-i)^2 = (x-1)(x^2-2ix-1)$
$= x(x^2-2ix-1) - 1(x^2-2ix-1)$
$= x^3 - 2ix^2 - x - x^2 + 2ix + 1$
$= x^3 - (1+2i)x^2 + (-1+2i)x + 1$

USING THE FACTOR THEOREM WHEN THERE ARE COMPLEX ROOTS

Recall that the factor theorem states:

- If a polynomial $p(x)$has a factor $(x – a)$, then $p(a) = 0$; and
- If $p(a) = 0$, then $(x – a)$ is a factor.

Example: Show that $x – i$, $x + 2i$ are both factors of $p(x) = 3x^3 – ix^2 + 10x – 8i$, and determine the other factors by inspection.

Answer: Use factor theorem to show that $x - i$ is a factor by substitute $x = i$;

$p(i) = 3i^3 - i \times i^2 + 10i - 8i$
$= -3i + i + 10i - 8i = 0$
$\therefore$ $(x-i)$ is a factor.

To show that $(x-2i)$ is a factor, substitute $x = -2i$
$p(-2i) = 3 \times (-2i)^3 - i \times (-2i)^2 + 10 \times (-2i) - 8i$
$= 3 \times -8i^3 - i \times 4i^2 - 20i - 8i$
$= 24i + 4i - 20i - 8i$
$= 0$
$\therefore$ $(x+2i)$ is a factor.

Thus, $3x^3 – ix^2 + 10x – 8i = (x-i)(x+2i)(ax+b)$
Equating the coefficient of x^3 term: $a = 3$
Equating the constant terms: $-8i = -i \times 2i \times b$
$-8i = 2b$
$-4i = b$
Therefore, the other factor is $(3x-4i)$

ACTIVITY 4A:

1. Factorise these equations:
   a) $16x^2 + 81$ L2
   b) $x^2 - 81$ L1
   c) $x^2 - 6x + 34$ L2

d) $4x^2 + 12x + 13$ L2

2. Solve the following complex equations giving your answer in exact form:

a) $z^2 - 8z + 28 = 0$ L3

b) $2z^2 + 6z + 11 = 0$ L3

c) $3z^2 - 4z = -9$ L3

3. Give the polynomial with the roots:

a) $2, 3i, -i$ L1

b) $4i, -4i$ L1

c) $i, -i, 1(multplicity\ 2)$ L1

4. Show that $x + 1$ and $x - 2i$ are both factors of $p(x) = x^3 + (4 - 2i)x^2 + (3 - 8i)x - 6i$ and determine the other factor. L2

5. Use Factor theorem to factorise $P(z) = z^3 + z^2(6 - i) + (8 - 6i)z - 8i$ L3

THE CONJUGATE ROOT THEOREM

Note the roots (1 + *i*) and (1 - *i*) are conjugate of each other. We call such roots **conjugate pairs**.

Any pair of conjugate roots gives a real polynomial

$$\boxed{[x-(a+ib)][x-(a-ib)] = x^2 - 2ax + (a^2 + b^2)}$$

The **conjugate root theorem** then asserts that:

The complex roots of any polynomial over the real numbers come in conjugate pairs.

OR

If p(x) is a polynomial over R with a complex root α , then $\overline{\alpha}$ is also a root.

OR

If $p(x) = 0$ then $p(\overline{x}) = 0$

Example

1. Give the polynomial (a) over C, and (b) over R, with roots 2, 3*i*.

Answer: a) $(x - 2)(x - 3i) = x^2 - x(2 + 3i) + 6i$ *no conjugate roots- complex polynomial*

b) $(x - 2)(x - 3i)(x + 3i) = (x - 2)(x^2 + 9)$

$= x^3 + 9x - 2x^2 - 18$

$= x^3 - 2x^2 + 9x - 18$ *conjugate pairs of roots – real polynomial*

2. Determine the roots of $2x^3 + 9x^2 + 8x - 39$, given that $-3 + 2i$ is a root. Hence factorise the polynomial.

Answer: From the conjugate – root theorem: if $x_1 = -3 + 2i$ is a root then $x_2 = -3 - 2i$ is also a root.

The third root will be real and correspond to a factor of the form, $ax + b$.

$2x^3 + 9x^2 + 8x - 39 = [x - (-3 + 2i)][x - (-3 - 2i)][ax + b]$

By inspection method: equate the coefficient of x^3: $a = 2$

Equate the constant term: $-(-3 + 2i) \times -(-3 - 2i) \times b = -39$

$$(-3 + 2i)(-3 - 2i)b = -39$$
$$(9 + 4)b = -39$$
$$b = -3$$

Therefore the third factor is $2x - 3$ and the root is $x_3 = \frac{3}{2}$

The three roots are $x_1 = -3 + 2i$, $x_2 = -3 - 2i$ and $x_3 = \frac{3}{2}$

The factors are: $2x^3 + 9x^2 + 8x - 39 = [x + 3 - 2i][x + 3 + 2i][2x - 3]$

SYMMETRY ROOTS

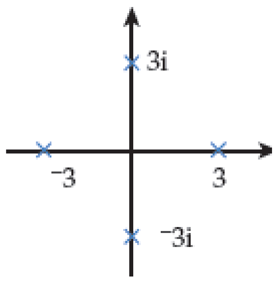
If $z^n = a$, then all n roots of z occurs at equal intervals around the origin. That is the angle between each roots is the same and is given by $\frac{2\pi}{n}$ or $\frac{360}{n}$ and will be of the same modulus.

That is if we have $z^4 = a$, we know that the four roots are 90° apart $\left(\frac{360}{4} = 90^o\right)$.

Example: Solve the equation $z^4 = 81$

Answer: modulus r $= \sqrt[4]{81} = 3$ and one solution is $z = 3$

The four roots are 90° apart $\left(\frac{360}{4} = 90^o\right)$ and the other solutions occur symmetrically about the origin at angles of 90° as shown in the diagram. Therefore the four solutions are: $3, 3i, -3, -3i$



ACTIVITY 4B:

1. $z = -4 + 5i$ is one solutions of the equation, $z^3 + Az^2 + 17z - 123 = 0$. Find the value of A and hence find the other two roots. L3

2. $\boldsymbol{z = 2 - 3i}$ is one of the roots of the polynomial equation $z^3 - pz^2 + qz - r = 0$, where p, q and r are real. Find the real root in terms of p or r L3

3. Solve these complex equations:

   a) $z^3 + 5z^2 + 17z + 13 = 0$ L3

   b) $2z^3 - 3z^2 + 18z + 10 = 0$ L3

4. Solve these equations and represent their solutions on the Argand diagram:

   a) $z^3 + 27i = 0$ L3

   b) $z^4 = -16$ L3

   c) $z^6 = 64$ L3

5. Find all the solutions of the equation $z^3 = mi$ where m is a positive real number. L4

LESSON ACTIVITY 5: DE MOIVRE'S THEOREM AND COMPLEX ROOTS

SUB-STRAND	SLO NO.	SPECIFIC LEARNING OUTCOMES	SKILL LEVEL	SLO CODE	ACHIEVED
1.3	12	Find roots of equations of the form $z^n = a + ib$, $z^n = r\ cis\ \theta$ where n is a positive integer (includes the use of de Moivre's theorem to solve equations and Argand diagrams to represent relationships between solutions). (This includes any roots from 9 – 11 above that are to be sketched on an Argand diagram.)	4	Cal1.3.4.2	

SOLVING EQUATIONS USING DE MOIVRE'S THEOREM.

COMPLEX ROOTS

We can use De Moivre's theorem to solve the types of equations of the form:

- $z^n = a$
- $z^n = rcis\theta$
- $z^n = a + bi$, where n is a counting number (integers) and a and b are real numbers.

These are the general steps of solving the types of equation in the form $z^n = a$:

1. **Write a in polar form as $rcis\,\theta$.**
2. **Write a more generally as $rcis(\theta + 2k\pi)$**
3. **Apply De Moivre's theorem to get** $r^{\frac{1}{n}}cis\left(\frac{\theta+2k\pi}{n}\right)$ **if working in radian, or** $r^{\frac{1}{n}}cis\left(\frac{\theta+360^o k}{n}\right)$ **if working in degree.**

Example

Solve the equation $z^3 = 8$ and show the solution on the Argand diagram.

Answer: $z^3 = 8$ *rectangular form*

1. express in polar form: modulus (r) $= \sqrt{8^2} = 8,\ \arg(\theta) = 0,$

$z^3 = 8cis(0)$ *polar form*

2. write in general form: $z^3 = 8cis(0 + 2k\pi)$ *general form*

3. Apply De Moivre's theorem: $z = (8cis(0 + 2k\pi))^{\frac{1}{3}}$

$$z = \sqrt[3]{8}cis\left(\frac{2k\pi}{3}\right)$$

$$z = 2cis\left(\frac{2k\pi}{3}\right) \text{ general solution } for\ k \in I$$

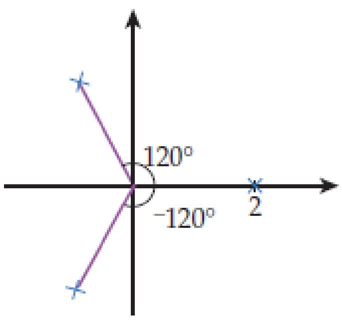
4. Substitute three consecutive integer values of $k = -1, 0, 1$ to obtain the solutions:
-

$$k = -1; \quad z = 2cis\left(\frac{2(-1)\pi}{3}\right) = 2cis\left(\frac{-2\pi}{3}\right)$$
$$= 2\left[cos\left(\frac{-2\pi}{3}\right) + isin\left(\frac{-2\pi}{3}\right)\right]$$
$$= 2\left[-cos\left(\frac{\pi}{3}\right) - isin\left(\frac{\pi}{3}\right)\right]$$
$$= 2\left[\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right]$$
$$= -1 - \sqrt{3}i$$

$$k = 0; \quad z = 2cis\left(\frac{2(0)\pi}{3}\right) = 2cos(0)$$
$$= 2[cos(0) + isin(0)]$$
$$= 2(1 + 0i) = 2$$

$$k = 1; \quad z = 2cis\left(\frac{2(1)\pi}{3}\right) = 2cos\left(\frac{2\pi}{3}\right)$$
$$= 2\left[cos\left(\frac{2\pi}{3}\right) + isin\left(\frac{2\pi}{3}\right)\right]$$
$$= 2\left[-cos\left(\frac{\pi}{3}\right) + isin\left(\frac{\pi}{3}\right)\right]$$
$$= 2\left[\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right]$$
$$= -1 + \sqrt{3}i$$

Graphing the solution on the Argand Diagram:



Example: Solve the equation $z^4 = 4 + 3i$ and express your answer in rectangular form.

Answer: $z^4 = 4 + 3i$ *rectangular form*

1. Express in polar form: modulus (r) $= \sqrt{4^2 + 3^2} = 5,\ \arg(\theta) = tan^{-1}\left(\frac{3}{4}\right) = 36.9°$

$$z^4 = 5cis(36.9° + 360k) \quad \textit{polar form}$$

2. Write in general form: $z^4 = 5cis(36.9° + 360k)$ *general form*

3. Apply De Moivre's theorem: $z = (5cis(36.9° + 360k))^{\frac{1}{4}}$

$$z = \sqrt[4]{5}cis\left(\frac{36.9°+360k}{4}\right)$$

$$z = 1.495cis(9.2° + 90k)$$ general solution $for\ k \in I$

4. Substitute three consecutive integer values of $k = -2, -1, 0, 1$ to obtain the solutions:

$k = -2;$ $z = 1.495cis(9.2° + 90(-2)) = 1.495cis(9.2° + (-180))$

$$= 1.495[cos(-170.8°) + isin(-170.8°)]$$
$$= -1.476 - 0.239i$$

$k = -1;$ $z = 1.495cis(9.2° + 90(-1)) = 1.495cis(9.2° + (-90))$

$$= 1.495[cos(-80.8°) + isin(-80.8°)]$$
$$= 0.239 - 1.476i$$

$k = 0;$ $z = 1.495cis(9.2° + 90(0)) = 1.495cis(9.2° + (0))$

$$= 1.495[cos(9.2°) + isin(9.2°)]$$
$$= 1.476 + 0.239i$$

$k = 1;$ $z = 1.495cis(9.2° + 90(1)) = 1.495cis(9.2° + (90))$

$$= 1.495[cos(99.2°) + isin(99.2°)]$$
$$= 0.239 - 1.495i$$

Example: Solve $z^4 = 16cis\left(\frac{\pi}{3}\right)$ over the complex numbers leaving your answer in polar form.

Answer: $z^4 = 16cis\left(\frac{\pi}{3}\right)$ *polar form*

2. Write in general form: $z^4 = 16cis\left(\frac{\pi}{3} + 2k\pi\right)$ *general form*

3. Apply De Moivre's theorem: $z = \left(16cis(\frac{\pi}{3} + 2k\pi)\right)^{\frac{1}{4}}$

$$z = \sqrt[4]{16}cis\left(\frac{\pi}{12} + \frac{k\pi}{2}\right)$$

$$z = 2cis\left(\left(\frac{\pi}{12} + \frac{k\pi}{2}\right)\right) \qquad \text{General solution } for\ k \in I$$

4. Substitute three consecutive integer values of $k = -2, -1, 0, 1$ to obtain the solutions:

$k = -2;\quad z = 2cis\left(\left(\frac{\pi}{12} + \frac{-2\pi}{2}\right)\right) = 2cis\left(\left(\frac{\pi}{12} - \frac{2\pi}{2}\right)\right) = 2cis\left(\left(-\frac{11\pi}{12}\right)\right)$

$k = -1;\quad z = 2cis\left(\left(\frac{\pi}{12} + \frac{-\pi}{2}\right)\right) = 2cis\left(\left(\frac{\pi}{12} - \frac{\pi}{2}\right)\right) = 2cis\left(\left(-\frac{5\pi}{12}\right)\right)$

$k = 0;\quad z = 2cis\left(\left(\frac{\pi}{12} + 0\right)\right) = 2cis\left(\left(\frac{\pi}{12}\right)\right)$

$k = 1;\quad z = 2cis\left(\left(\frac{\pi}{12} + \frac{\pi}{2}\right)\right) = 2cis\left(\left(\frac{7\pi}{12}\right)\right)$

Therefore solutions are: $z_1 = \left(-\frac{11\pi}{12}\right), z_2 = 2cis\left(-\frac{5\pi}{12}\right), z_3 = 2cis\left(\frac{\pi}{12}\right), z_4 = 2cis\left(\frac{7\pi}{12}\right)$

ACTIVITY 5A

1. If $\boldsymbol{z}^3 = 27cis\left(\frac{3\pi}{4}\right)$, find the roots and convert to rectangular form. L2
2. Find the roots of the equation $\boldsymbol{u}^2 = 2cis\frac{\pi}{4}$ L2
3. Find the values of $\boldsymbol{w}\ if\ \boldsymbol{w} = \sqrt[3]{x + i}$ L3
4. Find all the solutions for the equations $z^3 = 3 + 5i$ leaving your solutions in polar form.L3
5. If $\boldsymbol{z}^4 = -2 + 7i$, find the four complex roots in rectangular form and then plot them on the Argand diagram. L4
6. Solve the equations $\boldsymbol{z}^4 = 8\left(\sqrt{3} + i\right)$ by using De Moivre's theorem. Give answers in polar form. L3
7. Find in polar form, expressing in terms of n for all solutions of the equation $z^3 - 64n = 0$ where n is a positive real number. L4
8. Find all the solutions of $(z^2 + 1)^4 = 1$ where z is a complex number. L4