1 Investigate the effect of changing the mass (weight on the end) while the length of string is kept the same. Will this make the pendulum swing faster or slower?

Some weights you could use are:

- a key
- a sinker used for fishing
- a full can of soft-drink.

All of these have convenient points where you can attach a piece of string.

2 Investigate the effect of changing the length of string while the weight on the end is kept the same. Will this make the pendulum swing faster or slower?

You will need to make several accurate measurements:

- use a stop-watch for time
- use some accurate scales for the mass of the weight on the end
- use a ruler or tape-measure to record the length of the string.

If the pendulum swings too quickly to time one swing accurately, record the time for 10 swings and divide this time by 10 to get the time for one swing.

You could write the different times for a complete swing of the pendulum in a table like this one:

Pendulum swing times			
	Length 1 (.... cm)	Length 2 (.... cm)	Length 3 (.... cm)
Weight 1 (.... g)			
Weight 2 (.... g)			
Weight 3 (.... g)			

3 Write a short report on what you have noticed about the effect of changing the length and changing the mass on the period of the pendulum. Include some comments about why your investigation may not give very accurate results.

4 Try to write the relationship between some/all of the time, length and mass as a formula.

The rules of indices

Expressions with negative indices follow the same rules as ones with positive indices:

- **Multiplication** When *multiplying* numbers with the same base in power form, *add* the indices.
- **Division** When *dividing* numbers with the same base in power form, *subtract* the indices.
- **Powers of powers** When a base to a given index is raised to another index, we simplify by *multiplying* the indices.

Example

Simplify:

(a) $3x^{-2} \times 4x^{-3}$ (b) $(xy^2)^{-4}$

Write both answers with positive indices only.

Answer

(a) $3x^{-2} \times 4x^{-3} = 12x^{-5} = \dfrac{12}{x^5}$

(b) $(xy^2)^{-4} = x^{-4}y^{-8} = \dfrac{1}{x^4y^8}$

EXERCISE 9.3

1 Simplify these expressions. Leave all answers in power form.

(a) $x^{-4} \times x^2$

(b) $x^{-1} \times x^3$

(c) $\dfrac{x^{-10}}{x^{-5}}$

(d) $x^{-3} \div x^{-1}$

(e) $x^5 \div x^{-3}$

(f) $(x^{-3})^2$

(g) $(x^4)^{-1}$

(h) $4x^{-2} \times 5x^3$

(i) $\dfrac{-20x^{-2}}{4x}$

(j) $2x^3 \times 4x^{-2} \times 6x^{-1}$

2 Simplify these expressions. Write all answers with positive indices.

(a) x^2x^{-3}

(b) $\left(\dfrac{1}{x}\right)^{-2}$

(c) $\dfrac{x^{-4}}{x^{12}}$

(d) $(2x^{-1})^3$

(e) $(4x^2)^{-2}$

(f) $\dfrac{5x^2}{15x^3}$

(g) $\dfrac{4x^{-1}}{2x^{-3}}$

(h) $\dfrac{x^2y^{-4}}{x^3y^2}$

(i) $(x^{-1}y^2)^{-3}$

(j) $\left(\dfrac{3x}{4}\right)^{-1}$

(k) $\left(\dfrac{4x^2}{5}\right)^{-1}$

(l) $\left(\dfrac{x}{3}\right)^{-2}$

(m) $\left(\dfrac{2y}{x^2}\right)^{-2}$

(n) $\left(\dfrac{5x^3}{y^2}\right)^{-2}$

(o) $\left(\dfrac{4x^{-2}}{3y^5}\right)^{-1}$

(p) $\left(\dfrac{2a^2b^{-3}}{3a^{-1}b^5}\right)^{-4}$

3 Here are two different systems of numbering car number plates. Both systems use a total of 6 letters and numbers:

Last century: AA 0000 to ZZ 9999

This century: AAA 000 to ZZZ 999

(a) How many different letters are possible for the first position?

(b) How many different numbers are possible for the last position?

(c) Write, *in index form*, the total number of different number plates that are possible using last century's method.

(d) Write, *in index form*, the total number of different number plates that are possible using this century's method.

(e) Calculate the *difference* between the two numbers in (c) and (d).

Fractional indices

What does a fractional index mean? How do we interpret expressions such as $x^{\frac{1}{3}}$? What is the value of $64^{\frac{1}{2}}$? What meaning do we give to $8^{\frac{3}{2}}$?

Fractional indices are used to express *roots* of numbers. Recall that the square root of a number has the property that, when multiplied by itself, it gives the number concerned. For example:

$\sqrt{y} \times \sqrt{y} = y$ $\sqrt{5} \times \sqrt{5} = 5$ etc.

Now consider equivalent expressions with indices. For example:

$y^{\frac{1}{2}} \times y^{\frac{1}{2}} = y^1 = y$ (adding indices)

$5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 5^1 = 5$ (adding indices)

Clearly the power of $\frac{1}{2}$ is the same as 'square root'. In the same way, a power of $\frac{1}{3}$ gives 'cube roots', and so on. Collectively, square roots, cube roots, fourth roots, etc. are called **surds**.

In general: $a^{\frac{1}{x}} = \sqrt[x]{a}$

index form = surd form

Note that there is a special key for roots on a calculator. Depending on the make and model of the calculator, it will either be $\sqrt[x]{y}$ or $x^{\frac{1}{y}}$. It often needs the 'inverse' or 'F' key pressed first.

Example Write $x^{\frac{1}{5}}$ in surd form.

Answer $x^{\frac{1}{5}} = \sqrt[5]{x}$

Example Evaluate $64^{\frac{1}{3}}$.

Answer $64^{\frac{1}{3}} = \sqrt[3]{64} = 4$

Example Write $\sqrt[4]{x}$ in index form.

Answer $\sqrt[4]{x} = x^{\frac{1}{4}}$

Example Simplify $(64x^6)^{\frac{1}{2}}$.

Answer $(64x^6)^{\frac{1}{2}} = 8x^3$

Combining powers and roots

Powers and roots can be combined in the same index. For example $x^{\frac{2}{3}}$ means 'the cube root of x squared'.

In general: $x^{\frac{p}{q}} = \sqrt[q]{x^p}$

Example Evaluate $16^{\frac{3}{2}}$.

Answer $16^{\frac{3}{2}} = \left(16^{\frac{1}{2}}\right)^3 = 4^3 = 64$

Example Write $\sqrt{x^5}$ in index form.

Answer $\sqrt{x^5} = \left(x^5\right)^{\frac{1}{2}} = x^{\frac{5}{2}}$

(on multiplying the indices)

EXERCISE 9.4

1 Rewrite these expressions in surd form:

(a) $x^{\frac{1}{3}}$
(b) $y^{\frac{1}{6}}$
(c) $a^{\frac{1}{4}}$
(d) $5b^{\frac{1}{2}}$
(e) $2x^{\frac{1}{3}}$

2 Evaluate the following:

(a) $4^{\frac{1}{2}}$
(b) $16^{\frac{1}{4}}$
(c) $125^{\frac{1}{3}}$
(d) $64^{\frac{1}{3}}$
(e) $81^{\frac{1}{4}}$
(f) $36^{\frac{1}{2}}$
(g) $8^{\frac{1}{3}}$
(h) $256^{\frac{1}{2}}$
(i) $256^{\frac{1}{8}}$
(j) $\left(\frac{1}{4}\right)^{\frac{1}{2}}$
(k) $\left(\frac{9}{16}\right)^{\frac{1}{2}}$
(l) $\left(\frac{27}{64}\right)^{\frac{1}{3}}$
(m) $\left(\frac{81}{16}\right)^{\frac{1}{4}}$
(n) $\left(6\frac{1}{4}\right)^{\frac{1}{2}}$
(o) $(1.331)^{\frac{1}{3}}$

3 Rewrite these expressions in index form:

(a) $\sqrt[3]{x}$
(b) $\sqrt{y}$
(c) $\sqrt[5]{d}$
(d) $4\sqrt[4]{x}$
(e) $^{-}2\sqrt[6]{y}$

4 Rewrite these expressions in surd form:

(a) $x^{\frac{2}{3}}$
(b) $a^{\frac{3}{4}}$
(c) $3x^{\frac{2}{5}}$
(d) $2y^{\frac{3}{2}}$
(e) $5x^{\frac{5}{4}}$

5 Evaluate the following:

(a) $8^{\frac{2}{3}}$
(b) $32^{\frac{4}{5}}$
(c) $64^{\frac{2}{3}}$
(d) $36^{\frac{3}{2}}$
(e) $81^{\frac{3}{4}}$
(f) $49^{\frac{3}{2}}$
(g) $4^{\frac{5}{2}}$
(h) $125^{\frac{4}{3}}$
(i) $9^{\frac{3}{2}}$
(j) $16^{\frac{3}{4}}$
(k) $32^{\frac{2}{5}}$
(l) $8^{\frac{5}{3}}$

6 Evaluate the following. Give all answers as either fractions or mixed numbers.

(a) $\left(\frac{1}{4}\right)^{\frac{3}{2}}$ (e) $\left(\frac{27}{8}\right)^{\frac{4}{3}}$

(b) $\left(\frac{1}{27}\right)^{\frac{2}{3}}$ (f) $\left(\frac{243}{32}\right)^{\frac{3}{5}}$

(c) $\left(\frac{1}{16}\right)^{\frac{5}{4}}$ (g) $\left(\frac{125}{216}\right)^{\frac{4}{3}}$

(d) $\left(\frac{1}{8}\right)^{\frac{2}{3}}$ (h) $\left(6\frac{1}{4}\right)^{\frac{3}{2}}$

7 Rewrite these expressions in index form:

(a) $\sqrt[3]{x^5}$ (e) $\sqrt[3]{y^2}$

(b) $\sqrt{y^4}$ (f) $2\sqrt[4]{x^3}$

(c) $\sqrt{x^3}$ (g) $12\sqrt{x^4}$

(d) $\sqrt[4]{x^3}$ (h) $^{-}5\sqrt[3]{x^7}$

8 The temperature T (in degrees Celsius) of a cup of coffee decreases with time, t, according to the formula:

$$T = \frac{16t^{\frac{3}{2}} + 1000}{t^{\frac{3}{2}} + 10}$$

Calculate:

(a) the initial temperature

(b) the temperature when $t = 10$

(c) the temperature after a long time.

9 Initially a long-life insecticide sprayed on a ceiling kills 98% of all insects, and then the 'kill-rate' reduces by 5% each week.

The formula for the percentage killed after t weeks is $K = 98 \times (0.95)^t$.

(a) Copy and complete this table for the 'kill-rate' over the first five weeks:

Weeks (t)	1	2	3	4	5
Kill-rate					

(b) What percentage of insects are killed after 10 weeks?

(c) After how many weeks will fewer than half of the insects landing on the ceiling be killed?

10 Write down the last digit in each of these:

(a) 5^{100} (c) 7^{100}

(b) 6^{100} (d) $3^{100} + 2^{100}$

Fractional and negative indices

It is possible to use both negative indices and fractional indices at the same time. An approach which works well is to first deal with the negative index by taking the reciprocal of the base and changing the sign of the index at the same time.

Examples

Write $x^{\frac{-2}{3}}$ in surd form.

Answer

$$x^{\frac{-2}{3}} = \frac{1}{x^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{x^2}}$$

Example

Evaluate: (a) $625^{\frac{-1}{4}}$ (b) $\left(\frac{4}{9}\right)^{\frac{-3}{2}}$

Answer

(a) $625^{\frac{-1}{4}} = \frac{1}{625^{\frac{1}{4}}} = \frac{1}{\sqrt[4]{625}} = \frac{1}{5}$

(b) $\left(\frac{4}{9}\right)^{\frac{-3}{2}} = \left(\frac{9}{4}\right)^{\frac{3}{2}} = \left(\sqrt{\frac{9}{4}}\right)^3 = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$

EXERCISE 9.5

1 Rewrite these expressions in surd form:

(a) $x^{\frac{-1}{2}}$ (e) $4x^{\frac{-5}{4}}$

(b) $y^{\frac{-2}{3}}$ (f) $2y^{\frac{-3}{2}}$

(c) $a^{\frac{-1}{4}}$ (g) $5a^{\frac{-7}{3}}$

(d) $6a^{\frac{-4}{3}}$

2 Evaluate the following. Give all answers as fractions.

(a) $4^{-\frac{1}{2}}$ (f) $16^{-\frac{3}{4}}$ (k) $64^{-\frac{2}{3}}$

(b) $36^{-\frac{1}{2}}$ (g) $16^{-\frac{3}{2}}$ (l) $32^{-\frac{4}{5}}$

(c) $81^{-\frac{1}{2}}$ (h) $64^{-\frac{5}{6}}$ (m) $243^{-\frac{2}{5}}$

(d) $64^{-\frac{1}{3}}$ (i) $9^{-\frac{5}{2}}$ (n) $256^{-\frac{3}{8}}$

(e) $27^{-\frac{2}{3}}$ (j) $100^{-\frac{1}{2}}$

3 Evaluate the following. Give all answers as either whole numbers, fractions or mixed numbers.

(a) $\left(\frac{1}{16}\right)^{-\frac{1}{4}}$ (i) $\left(\frac{125}{216}\right)^{-\frac{1}{3}}$

(b) $\left(\frac{1}{25}\right)^{-\frac{1}{2}}$ (j) $\left(\frac{9}{16}\right)^{-\frac{5}{2}}$

(c) $\left(\frac{9}{64}\right)^{-\frac{1}{2}}$ (k) $\left(\frac{4}{9}\right)^{-\frac{3}{2}}$

(d) $\left(\frac{1}{125}\right)^{-\frac{1}{3}}$ (l) $\left(\frac{36}{25}\right)^{-\frac{3}{2}}$

(e) $\left(\frac{1}{9}\right)^{-\frac{3}{2}}$ (m) $\left(\frac{36}{49}\right)^{-\frac{3}{2}}$

(f) $\left(\frac{1}{16}\right)^{-\frac{3}{4}}$ (n) $\left(1\frac{7}{9}\right)^{-\frac{3}{2}}$

(g) $\left(\frac{1}{49}\right)^{-\frac{1}{2}}$ (o) $\left(12\frac{1}{4}\right)^{-\frac{1}{2}}$

(h) $\left(\frac{125}{8}\right)^{-\frac{2}{3}}$ (p) $\left(\frac{1000}{343}\right)^{-\frac{2}{3}}$

4 Simplify the following, using surd form:

(a) $\dfrac{1}{x^{-\frac{1}{4}}}$ (d) $\dfrac{1}{2x^{-\frac{1}{5}}}$

(b) $\left(\dfrac{9}{4x}\right)^{-\frac{3}{2}}$ (e) $\dfrac{1}{8x^{-\frac{1}{3}}}$

(c) $\dfrac{16}{x^{-\frac{1}{2}}}$ (f) $\dfrac{3}{x^{-\frac{2}{3}}}$

(g) $\dfrac{5x^{-\frac{5}{3}}}{2}$ (h) $\dfrac{27}{64x^{-\frac{1}{3}}}$

5 Rewrite these expressions in index form:

(a) $\dfrac{1}{\sqrt{x^3}}$ (e) $\dfrac{2}{\sqrt{x^3}}$

(b) $\dfrac{1}{\sqrt[3]{x^4}}$ (f) $\dfrac{4}{\sqrt{x^5}}$

(c) $\dfrac{1}{\sqrt[5]{x^2}}$ (g) $\dfrac{1}{3\sqrt{x}}$

(d) $\dfrac{1}{\sqrt[4]{x^3}}$ (h) $\dfrac{7}{4\sqrt[3]{x^5}}$

6 Simplify $\left(4a^2\right)^{-\frac{1}{2}}$.

7 Use the rules of indices to simplify the following expressions:

(a) $x^{n+1} \times x^{n-2}$ (i) $\dfrac{x^{\frac{2}{5}}}{x^{\frac{1}{3}} \times x^{\frac{1}{5}}}$

(b) $x^{4n} \div x^n$ (j) $\left(x^{\frac{1}{4}}y^{\frac{1}{3}}\right)^{-12}$

(c) $\dfrac{x^{3n} \times x^{2n}}{x^{4n}}$ (k) $\left(2x^{-\frac{1}{2}}\right)^4$

(d) $\dfrac{x^{7n} \times x^{2n}}{x^{9n}}$ (l) $\left(3x^{\frac{1}{3}}\right)^{-3}$

(e) $\dfrac{x^{-2} \times x^{-1}}{x^4 \times x^{-6}}$ (m) $\left(2x^{\frac{1}{2}}y^{-\frac{1}{4}}\right)^{-4}$

(f) $\dfrac{x^{\frac{1}{5}} \times x^{\frac{2}{5}}}{x^{\frac{4}{5}}}$ (n) $\left(3x^{\frac{3}{2}}y^{\frac{3}{4}}\right)^{-2}$

(g) $\dfrac{x^{-\frac{1}{2}} \times x^{\frac{1}{4}}}{x^{\frac{3}{8}} \times x^2}$ (o) $\left(\dfrac{x^{\frac{1}{2}}y^{\frac{1}{3}}}{z^{\frac{3}{5}}}\right)^{-30}$

(h) $\dfrac{x^{\frac{1}{3}} \times x^{\frac{1}{6}}}{x^{\frac{3}{4}}}$

8 Simplify $\left(27a^{\frac{3}{4}}\right)^{\frac{1}{3}} \times \left(9a^{-\frac{3}{2}}\right)^{-\frac{1}{2}}$

9 Calculate the value of $\left(-a\right)^{-\frac{2}{3}}$ when $a = \frac{1}{64}$.

Expressing numbers as powers of the same base

Many numbers can be written quite simply as a power of another number. For example, 9 can be written as a power of 3—i.e. 3^2. And $\frac{1}{27}$ can also be written as a power of 3—i.e. 3^{-3}.

A useful approach when simplifying expressions where some bases are powers of another base is to express all terms as powers of the *smaller* number.

Example

Simplify $\frac{4^{3n}}{2^{5n}}$.

Answer

First note that $4 = 2^2$. Therefore replace 4 by its equivalent 2^2.

$$\frac{4^{3n}}{2^{5n}} = \frac{(2^2)^{3n}}{2^{5n}} = \frac{2^{6n}}{2^{5n}} = 2^{6n-5n} = 2^n$$

EXERCISE 9.6

1 Express the following as powers of 2:
(a) 8
(b) 32
(c) $\frac{1}{2}$
(d) $\frac{1}{4}$
(e) 1

2 Express the following as powers of 4:
(a) 2
(b) 8
(c) $\frac{1}{4}$
(d) $\frac{1}{16}$
(e) $\frac{1}{32}$

3 Express the following as powers of 9:
(a) 27
(b) 3
(c) $\frac{1}{3}$
(d) $\frac{1}{81}$
(e) $\sqrt{3}$

4 Express the following as powers of 5:
(a) 25
(b) 625
(c) 5
(d) 1
(e) 0.2
(f) $\frac{1}{125}$

5 Express the following as powers of $\frac{1}{3}$:
(a) 3
(b) 9
(c) $\sqrt{3}$
(d) $\frac{1}{27}$
(e) 243

6 Solve these equations:
(a) $3^x = 81$
(b) $5^{-x} = 25$
(c) $2^{x+1} = 8^{x-4}$
(d) $8^{x-1} = 16^{x-2}$
(e) $x^{\frac{1}{2}} = 5$
(f) $3x^{\frac{1}{2}} - 1 = 20$

7 Given that $8^n = 2^p$, write down an expression for p in terms of n.

8 Simplify the following expressions by changing some powers to the same base:
(a) $\frac{8^{4n}}{16^{2n}}$
(b) $\frac{9^{4n}}{81^{2n}}$
(c) $\frac{64^n}{4^n}$
(d) $\frac{8^n \times 16^{2n}}{256^n \times 64^{2n}}$
(e) $\frac{3^{n+2} \times 27^{n-1}}{3^{n-2} \times 27^{n+1}}$
(f) $\frac{5^{2n} \times 125^{n-1}}{625^{n+2}}$
(g) $\frac{6^{2n+1}}{4^{n+2} \times 3^{2n-1}}$ 买
(h) $\frac{2^{2n+1} \times 4^{3n-2}}{8^{n+4} \times 16^{2n+1}}$
(i) $\frac{32^{\frac{n}{5}} \times 8^{n-3}}{128^{n+1} \times 4^{3n}}$
(j) $\frac{81^{\frac{3n}{4}} \times 27^{\frac{n}{3}}}{243^n \times 3^{2-4n}}$
(k) $\frac{(3 \times 2^n) \times (3 \times 8^n)}{3 \times 16^{1-n}}$

10 Surds

Square roots and cube roots

The **square root** of a number is defined to be the number that, when squared (or multiplied by itself) gives the original number.

We use the symbol $\sqrt{\ }$ to show square root.

In a similar way, the **cube root** of a number is defined to be the number that, when cubed (or multiplied by itself by itself) gives the original number.

We use the symbol $\sqrt[3]{\ }$ to show cube root.

Examples

$\sqrt{49} = 7$ because $7 \times 7 = 7^2 = 49$

$\sqrt[3]{125} = 5$ because $5 \times 5 \times 5 = 5^3 = 125$

EXERCISE 10.1

1 Simplify these square root expressions:

(a) $\sqrt{16} \times \sqrt{16}$ (d) $\sqrt{25} \times \sqrt{25}$

(b) $\sqrt{2} \times \sqrt{2}$ (e) $\left(\sqrt{9}\right)^2$

(c) $\sqrt{13} \times \sqrt{13}$ (f) $\left(\sqrt{5}\right)^2$

2 Simplify these cube root expressions:

(a) $\sqrt[3]{64} \times \sqrt[3]{64} \times \sqrt[3]{64}$

(b) $\sqrt[3]{343} \times \sqrt[3]{343} \times \sqrt[3]{343}$

(c) $\sqrt[3]{0.8} \times \sqrt[3]{0.8} \times \sqrt[3]{0.8}$

3 Evaluate:

(a) $(\sqrt{25})^3$ (b) $\sqrt{25^3}$

4 Evaluate:

(a) $\left(\sqrt[3]{512}\right)^2$ (b) $\sqrt[3]{512^2}$

5 Calculate the value of these root expressions:

(a) $\sqrt{4096}$ (c) $\sqrt{\sqrt[3]{4096}}$

(b) $\sqrt[3]{4096}$ (d) $\sqrt[3]{\sqrt{4096}}$

6 Calculate the value of the following 'nested' square roots:

(a) $\sqrt{43\ 046\ 721}$

(b) $\sqrt{\sqrt{43\ 046\ 721}}$

(c) $\sqrt{\sqrt{\sqrt{43\ 046\ 721}}}$

(d) $\sqrt{\sqrt{\sqrt{\sqrt{43\ 046\ 721}}}}$

7 Explain what happens to any positive number when you keep taking the square root.

8 Calculate the value of the following 'nested' cube roots:

(a) $\sqrt[3]{10\ 077\ 696}$

(b) $\sqrt[3]{\sqrt[3]{10\ 077\ 696}}$

(c) $\sqrt[3]{^{-}134\ 217\ 728}$

(d) $\sqrt[3]{\sqrt[3]{^{-}134\ 217\ 728}}$

(e) $\sqrt[3]{\sqrt[3]{\sqrt[3]{^{-}134\ 217\ 728}}}$

9 Explain why negative numbers cannot have a square root but do have a cube root.

Surds

- Use a calculator to evaluate $(1.414\,213\,562)^2$. What do you get?
- Evaluate $\sqrt{2}$.
- Is the value for $\sqrt{2}$ on your calculator the exact value for the square root of 2?

Rational numbers are those which can be written as the ratio of two integers. In other words, they can be written in fraction form. However, not all roots (i.e. square roots, cube roots, etc.) of rational

numbers are themselves rational numbers. A **surd** is an irrational root of a rational number. Here are some examples of surds:

$\sqrt{2}$ $\sqrt{5}$ $\sqrt{12}$ $\sqrt[3]{4}$ $\sqrt[3]{50}$

The following are not surds:

$\sqrt{9}$ $\sqrt[3]{64}$ $\sqrt{2.25}$

because each has an exact rational number value, i.e. 3, 4 and 1.5, respectively.

Although surds do not have exact rational number values, they can be simplified in several ways, and follow certain mathematical rules.

We initially restrict the study of surds to square roots, and talk about 'root' 6 for example, rather than 'square root' 6.

Simplifying surds

If the number under a root sign is a perfect square, the root can be evaluated—e.g. $\sqrt{81} = 9$.

For other numbers it is still possible to simplify surds, if the number under the root sign is a multiple of a perfect square. Recall that perfect squares are numbers such as 4, 9, 16, 25, 36, 49, 64, 81, etc.

We always try to simplify surds so that we have the lowest possible whole number under the root sign. The method involves writing the number under the surd as a product of a perfect square and another number.

Example Simplify $\sqrt{45}$.

Answer $\sqrt{45} = \sqrt{9 \times 5} = \sqrt{9} \times \sqrt{5} = 3\sqrt{5}$

Example Simplify $3\sqrt{40}$.

Answer

$$\begin{aligned} 3\sqrt{40} &= 3 \times \sqrt{4 \times 10} \\ &= 3 \times \sqrt{4} \times \sqrt{10} \\ &= 3 \times 2 \times \sqrt{10} \\ &= 6\sqrt{10} \end{aligned}$$

Complete surds

To reverse the process of 'simplifying' we can write **complete surds**. This is when the expression is entirely under the root sign.

Example

Write $5\sqrt{7}$ as a complete surd.

Answer

First express 5 as the square root of a number—i.e. $5 = \sqrt{25}$

$$\begin{aligned} 5\sqrt{7} &= \sqrt{25} \times \sqrt{7} \\ &= \sqrt{25 \times 7} && \text{(multiplying the numbers under the root signs)} \\ &= \sqrt{175} \end{aligned}$$

EXERCISE 10.2

1 Simplify these surds to their lowest form:

(a) $\sqrt{20}$
(b) $\sqrt{12}$
(c) $\sqrt{72}$
(d) $\sqrt{18}$
(e) $\sqrt{80}$
(f) $\sqrt{48}$
(g) $\sqrt{8}$
(h) $\sqrt{75}$
(i) $\sqrt{44}$
(j) $\sqrt{24}$
(k) $\sqrt{128}$
(l) $\sqrt{27}$
(m) $\sqrt{28}$
(n) $\sqrt{50}$
(o) $\sqrt{32}$
(p) $\sqrt{432}$
(q) $\sqrt{245}$
(r) $\sqrt{125}$
(s) $\sqrt{147}$
(t) $\sqrt{300}$
(u) $\sqrt{405}$
(v) $\sqrt{242}$
(w) $\sqrt{1575}$
(x) $\sqrt{5000}$

2 Simplify these expressions by writing them as surds in their lowest form:

(a) $2\sqrt{12}$
(b) $5\sqrt{8}$
(c) $4\sqrt{24}$
(d) $3\sqrt{8}$
(e) $7\sqrt{50}$
(f) $4\sqrt{99}$
(g) $2\sqrt{27}$
(h) $4\sqrt{12}$
(i) $5\sqrt{20}$
(j) $4\sqrt{125}$
(k) $3\sqrt{18}$
(l) $2\sqrt{75}$
(m) $5\sqrt{18}$

3 Write these expressions as multiples of surds in their lowest form:

(a) $\sqrt[3]{81}$ (b) $\sqrt[4]{32}$ (c) $\sqrt[3]{16}$ (d) $\sqrt[5]{64}$ (e) $\sqrt[3]{1250}$ (f) $\sqrt[3]{4000}$

4 Write these as complete surds:

(a) $2\sqrt{3}$ (b) $6\sqrt{2}$ (c) $7\sqrt{5}$ (d) $3\sqrt{6}$ (e) $2\sqrt{7}$ (f) $3\sqrt{11}$ (g) $4\sqrt{3}$ (h) $2\sqrt{6}$ (i) $5\sqrt{3}$ (j) $7\sqrt{7}$ (k) $9\sqrt{2}$ (l) $10\sqrt{2}$

5 Write these as complete surds:

(a) $2\sqrt[3]{5}$ (b) $3\sqrt[4]{6}$ (c) $5\sqrt[3]{4}$ (d) $4\sqrt[5]{2}$ (e) $10\sqrt[3]{6}$

6 Given that $\sqrt{5} \approx 2.236$, find the approximate values of the following *without using the square root key* on a calculator:

(a) $2\sqrt{5}$ (b) $\sqrt{20}$ (c) $\sqrt{45} + \sqrt{80}$ (d) $\sqrt{500}$ (e) $\sqrt{\dfrac{1}{5}}$

7 Simplify these expressions by looking for common factors:

(a) $\dfrac{\sqrt{6}}{\sqrt{2}}$ (b) $\dfrac{\sqrt{5}}{\sqrt{20}}$ (c) $\dfrac{2\sqrt{8}}{3\sqrt{12}}$ (d) $\dfrac{5\sqrt{20}}{3\sqrt{8}}$ (e) $\dfrac{2\sqrt{10}}{\sqrt{125}}$ (f) $\dfrac{3\sqrt{15}}{2\sqrt{10}}$

(g) $\dfrac{8\sqrt{6}}{10\sqrt{12}}$ (h) $\dfrac{4\sqrt{24}}{3\sqrt{18}}$ (i) $\dfrac{6\sqrt{6}}{2\sqrt{8}}$ (j) $\dfrac{5\sqrt{24}}{2\sqrt{50}}$ (k) $\dfrac{8 + \sqrt{20}}{4}$ (l) $\dfrac{4 + \sqrt{28}}{2}$

(m) $\dfrac{\sqrt{10} + \sqrt{50}}{\sqrt{5}}$ (n) $\dfrac{3 + \sqrt{27}}{3}$ (o) $\dfrac{\sqrt{12} + \sqrt{20}}{4}$ (p) $\dfrac{\sqrt{8} + \sqrt{128}}{\sqrt{2}}$

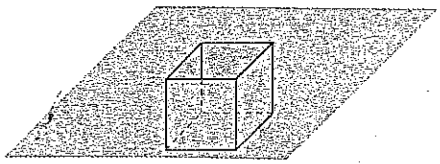
EXERCISE 10.3 Applications

1 A flag-maker sells square flags. The area of each flag is 5 m^2.

(a) What is the side length? Leave your answer in surd form.

(b) Calculate the side length to the nearest cm.

2 A landscape gardener prepares a hole for planting a large date palm. The hole is in the shape of a cube, and 4 m^3 of soil has been removed altogether.

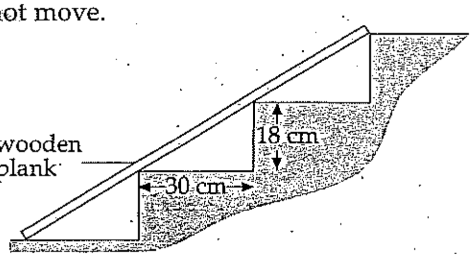


Estimate the depth of the hole:

(a) expressing your answer in surd form

(b) writing your answer to the nearest cm.

3 Two levels of a building site are linked by three steps. Each step is 30 cm wide by 18 cm high. In order to move a wheelbarrow between the two levels, a plank is placed on top of the steps, and fastened so that it does not move.

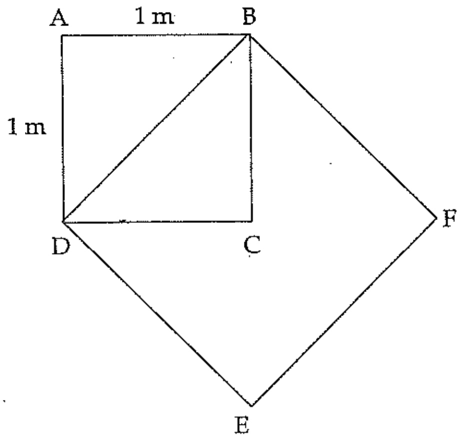


Express the length of the plank as a surd in its simplest form.

Mathematicians in ancient Greece studied various problems such as 'doubling the square' and 'doubling the cube'.

4 Doubling the square

ABCD and BDEF are squares. The area of square ABCD is 1 m^2.



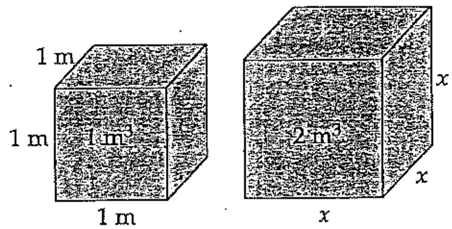
(a) Express the length of BD in surd form.

(b) Explain why the area of BDEF is 2 m^2.

5 Doubling the cube

The inspiration for this problem may have come from Plato (Athens, 430 BC):

Eratosthenes in his work entitled Platonicus *relates that when the god proclaimed to the Delians through the oracle that in order to get rid of a plague, they should construct an altar double that of the existing cubic one, their craftsmen fell into great perplexity in their efforts to discover how a solid could be made the double of a similar solid; they therefore went to ask Plato about it, and he replied that the oracle meant, not that the god wanted an altar of double the size, but that he wished, in setting them the task, to shame the Greeks for their neglect of mathematics and their contempt of geometry.*



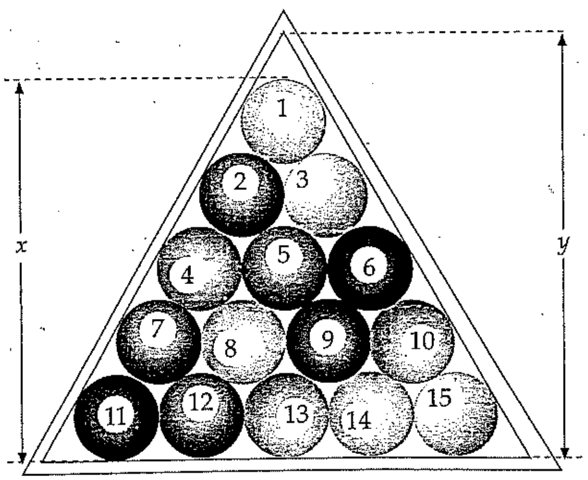
Suppose the altar was a cubic stone measuring exactly 1 m by 1 m by 1 m, so had a volume of 1 m^3.

The problem then can be expressed as finding the side length of a cube with volume of 2 m^3.

(a) What is this length, expressed as a surd?

(b) Calculate the length correct to 3 sf.

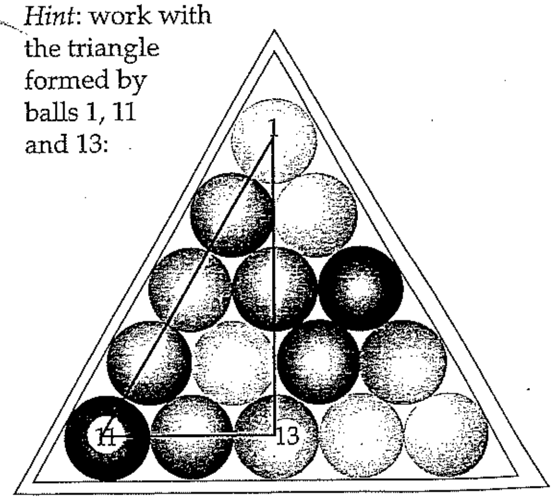
6 A game of snooker starts by placing 15 identical balls in a wooden triangular frame.



Each ball has a radius of 2 cm.

(a) Express the height of the triangle of 15 balls in surd form (the length marked x).

Hint: work with the triangle formed by balls 1, 11 and 13:

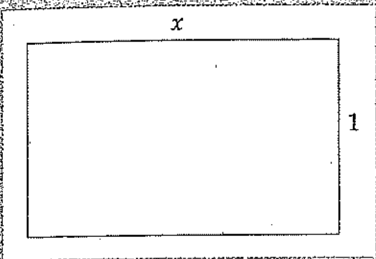


(b) Express the height (y) of the inside of the wooden frame in surd form.

INVESTIGATION – The Golden Ratio

The Golden Ratio was known to the ancient Greeks, and used extensively in their art and architecture. It can also be seen in the paintings of Leonardo da Vinci.

It is defined in a rectangle as follows:



The ratio of the long side to the short side
is the same as
the sum of the long and short sides to the long side

If the longer side is x and the shorter side is 1, then we have:

$$x:1 = (x+1):x$$

$$\frac{x}{1} = \frac{x+1}{x}$$

$$x^2 = x+1$$

$$x^2 - x - 1 = 0$$

1 Use the quadratic formula to work out the Golden Ratio from this equation ($x^2 - x - 1 = 0$).

$$x = \frac{{}^-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Give your answer in two ways:

(a) expressed using surds

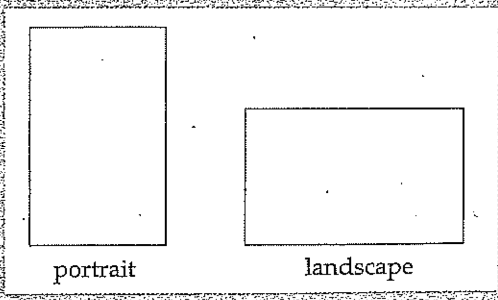
(b) as a decimal.

2 There are two different ways of displaying pictures—as landscape or portrait.

A picture-framer makes rectangular frames 70 cm wide. The sides fit the Golden Ratio. Calculate the height of the frames in each case:

(a) to be used as portrait

(b) to be used as landscape.



Adding/subtracting surds

The key to adding/subtracting surds is to look for 'like' terms. These are ones in which the number under the root sign is the same.

Example

Simplify $2\sqrt{2} + 4\sqrt{3} - 5\sqrt{2} + 2\sqrt{3}$.

Answer

$2\sqrt{2} + 4\sqrt{3} - 5\sqrt{2} + 2\sqrt{3} = {}^{-}3\sqrt{2} + 6\sqrt{3}$

No further simplification is possible.

EXERCISE 10.4

1 Add/subtract these surds:

(a) $\sqrt{5} + 3\sqrt{5} + 6\sqrt{5}$
(b) $2\sqrt{3} + 5\sqrt{3} + 6\sqrt{3}$
(c) $5\sqrt{2} + 2\sqrt{2} + 8\sqrt{2}$
(d) $9\sqrt{3} - 6\sqrt{3}$
(e) $2\sqrt{5} - \sqrt{5}$
(f) $4\sqrt{7} + 3\sqrt{7} - 14\sqrt{7}$
(g) $8\sqrt{3} - 4\sqrt{3} + 6\sqrt{3}$
(h) $5\sqrt{2} + 7\sqrt{2} - 4\sqrt{2} + 5\sqrt{2}$
(i) $6\sqrt{3} + 2\sqrt{5} - 2\sqrt{3} + 4\sqrt{5}$
(j) $3\sqrt{7} - \sqrt{2} + 4\sqrt{2} + 9\sqrt{2}$
(k) $6\sqrt{3} + 8\sqrt{3} - 6\sqrt{2}$
(l) $4\sqrt{3} + 5\sqrt{2} - 6\sqrt{2} - 3\sqrt{3}$
(m) $4\sqrt{2} + 5\sqrt{5} + 11\sqrt{2} - 6\sqrt{5}$

2 Simplify these surd expressions, and then simplify further by adding or subtracting:

(a) $\sqrt{8} + \sqrt{32}$
(b) $\sqrt{12} + \sqrt{75}$
(c) $\sqrt{48} - \sqrt{27}$
(d) $\sqrt{18} + \sqrt{50} - \sqrt{32}$
(e) $\sqrt{45} - \sqrt{20}$
(f) $\sqrt{28} + \sqrt{63} - \sqrt{112}$

3 Simplify first, and then add or subtract:

(a) $\sqrt{50} - \sqrt{72} + \sqrt{18} - \sqrt{32}$
(b) $\sqrt{80} - \sqrt{5} + \sqrt{45} - \sqrt{180}$
(c) $\sqrt{8} + \sqrt{75} - \sqrt{72} - \sqrt{3}$
(d) $\sqrt{27} + \sqrt{48} + \sqrt{75} - \sqrt{192}$
(e) $\sqrt{108} - \sqrt{12} + \sqrt{128} + \sqrt{200}$
(f) $\sqrt{63} + \sqrt{20} - \sqrt{125} + \sqrt{28}$
(g) $\sqrt{75} - \sqrt{80} + \sqrt{27} - \sqrt{20}$
(h) $\sqrt{24} + \sqrt{27} + \sqrt{216} - \sqrt{75}$
(i) $\sqrt{50} - \sqrt{108} + \sqrt{98} + \sqrt{147}$
(j) $\sqrt{48} + \sqrt{75} - \sqrt{108} + \sqrt{20}$
(k) $\sqrt{98} - \sqrt{192} + \sqrt{50} - \sqrt{108}$
(l) $\sqrt{80} + \sqrt{18} - \sqrt{180} - \sqrt{8}$
(m) $\sqrt{27} + \sqrt{8} + \sqrt{48} - \sqrt{128}$

Multiplying surds

When multiplying surds, just multiply the numbers under the root signs.

Example

Simplify $\sqrt{5} \times \sqrt{3}$.

Answer

$\sqrt{5} \times \sqrt{3} = \sqrt{5 \times 3} = \sqrt{15}$

Example

Simplify $3\sqrt{2} \times 5\sqrt{8}$.

Answer

$$\begin{aligned} 3\sqrt{2} \times 5\sqrt{8} &= 15 \times \sqrt{2 \times 8} \\ &= 15\sqrt{16} \\ &= 15 \times 4 \\ &= 60 \end{aligned}$$

INVESTIGATION – Surds and trigonometry

Sin, cos and tan of angles such as 30°, 45° and 60° can be expressed in surd form.

Remember

$$\sin = \frac{\text{opp}}{\text{hyp}}, \quad \cos = \frac{\text{adj}}{\text{hyp}}, \quad \tan = \frac{\text{opp}}{\text{adj}}$$

OR $s = h\sin A,\ c = h\cos A,\ s = c\tan A$.

1 Evaluate $\frac{1}{\sqrt{2}}$.

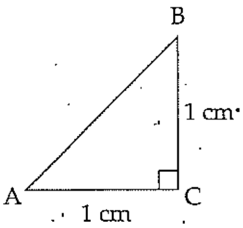
2 Evaluate $\sin 45°$.

3 True or false? $\sin 45° = \frac{1}{\sqrt{2}}$.

4 Explain why angle A is 45°.

5 Explain why AB = $\sqrt{2}$ cm.

6 Express cos 45° in surd form.

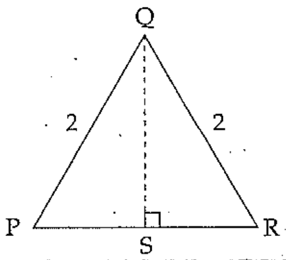


The diagram shows an equilateral triangle PQR. QS ⊥ PR.

7 Write down the values of these angles and lengths:

(a) ∠QRP (c) SR

(b) ∠SQR (d) QS.



8 Write down the value of sin 30° as a fraction.

9 Copy and complete this table. All the values should be whole numbers or fractions, using surds if necessary.

	30°	45°	60°
sin			
cos			
tan			

xpansion of surd expressions

ackets with surds in them can be expanded in the usual way.
ok for simplifications.

xample

pand and simplify $\sqrt{6}\left(\sqrt{3}+4\right)$.

Answer

$$\sqrt{6}\left(\sqrt{3}+4\right)=\sqrt{18}+4\sqrt{6}$$
$$=3\sqrt{2}+4\sqrt{6}$$

xample

pand and simplify $\left(2\sqrt{2}+4\sqrt{3}\right)\left(6\sqrt{2}-5\sqrt{3}\right)$.

nswer

$$\sqrt{2}+4\sqrt{3}\big)\left(6\sqrt{2}-5\sqrt{3}\right)=12\times\sqrt{2}\times\sqrt{2}-10\times\sqrt{2}\times\sqrt{3}+24\times\sqrt{3}\times\sqrt{2}-20\times\sqrt{3}\times\sqrt{3}$$
$$=12\sqrt{4}-10\sqrt{6}+24\sqrt{6}-20\sqrt{9}$$
$$=12\times2-10\sqrt{6}+24\sqrt{6}-20\times3$$
$$=24+14\sqrt{6}-60$$
$$=14\sqrt{6}-36$$

XERCISE 10.5

Multiply these surds, and simplify the final answer if possible:

(a) $\sqrt{2}\times\sqrt{5}$
(b) $\sqrt{7}\times\sqrt{3}$
(c) $\sqrt{5}\times\sqrt{3}$
(d) $\sqrt{6}\times\sqrt{5}$
(e) $\sqrt{2}\times\sqrt{7}$
(f) $\sqrt{3}\times\sqrt{6}$
(g) $\sqrt{2}\times\sqrt{6}$
(h) $\sqrt{7}\times\sqrt{14}$
(i) $\sqrt{10}\times\sqrt{5}$
(j) $\sqrt{11}\times\sqrt{22}$
(k) $\sqrt{6}\times\sqrt{6}$
(l) $\sqrt{3}\times\sqrt{3}$
(m) $\left(\sqrt{7}\right)^2$
(n) $\left(\sqrt{11}\right)^2$
(o) $\sqrt{18}\times\sqrt{24}$
(p) $\sqrt{12}\times\sqrt{32}$
(q) $\sqrt{8}\times\sqrt{22}$
(r) $\sqrt{48}\times\sqrt{12}$

2 (a) Evaluate (to 4 sf):
(i) $\sqrt{6}\times\sqrt{2}$
(ii) $\sqrt{12}$

(b) Write $\sqrt{a}\times\sqrt{b}$ as a complete surd.

3 Multiply these surd expressions, and simplify if possible:

(a) $2\sqrt{3}\times5\sqrt{2}$
(b) $4\sqrt{2}\times3\sqrt{7}$
(c) $5\sqrt{6}\times2\sqrt{3}$
(d) $4\sqrt{5}\times\sqrt{3}$
(e) $\sqrt{14}\times2\sqrt{7}$
(f) $3\sqrt{2}\times\sqrt{6}$
(g) $2\sqrt{3}\times\sqrt{5}$
(h) $4\sqrt{10}\times\sqrt{15}$
(i) $2\sqrt{5}\times3\sqrt{15}$
(j) $5\sqrt{6}\times2\sqrt{15}$
(k) $3\sqrt{21}\times5\sqrt{28}$
(l) $\left(4\sqrt{5}\right)^2$
(m) $\left(3\sqrt{3}\right)^2$
(n) $\left({}^-6\sqrt{3}\right)^2$
(o) $\left({}^-3\sqrt{7}\right)^2$

4 Simplify these surd expressions first, and then multiply:

(a) $\sqrt{18}\times\sqrt{12}$
(b) $\sqrt{12}\times\sqrt{8}$
(c) $\sqrt{27}\times\sqrt{18}$
(d) $\sqrt{25}\times\sqrt{18}$
(e) $\sqrt{50}\times\sqrt{10}$
(f) $\sqrt{48}\times\sqrt{12}$
(g) $\sqrt{8}\times\sqrt{20}$
(h) $\sqrt{18}\times\sqrt{24}$
(i) $\sqrt{125}\times\sqrt{48}$
(j) $\sqrt{24}\times\sqrt{54}$
(k) $\sqrt{300}\times\sqrt{500}$
(l) $\sqrt{1000}\times\sqrt{80}$

5 Expand these expressions, leaving all surds in their simplest form:

(a) $\sqrt{2}\left(\sqrt{3}+4\right)$
(b) $\sqrt{3}\left(1-\sqrt{2}\right)$
(c) $\sqrt{5}\left(\sqrt{2}+\sqrt{3}\right)$
(d) $\sqrt{6}\left(4-\sqrt{3}\right)$
(e) $\sqrt{3}\left(1+\sqrt{5}\right)$
(f) $\sqrt{2}\left(\sqrt{7}-\sqrt{3}\right)$
(g) $\sqrt{5}\left(4+\sqrt{8}\right)$
(h) $\sqrt{11}\left(4+\sqrt{2}\right)$
(i) $\sqrt{2}\left(\sqrt{5}+\sqrt{6}\right)$
(j) $\sqrt{3}\left(\sqrt{7}-\sqrt{10}\right)$
(k) $\sqrt{2}\left(\sqrt{15}-\sqrt{6}\right)$
(l) $\sqrt{5}\left(\sqrt{10}+\sqrt{15}\right)$
(m) $\sqrt{2}\left(\sqrt{6}-\sqrt{2}\right)$
(n) $\sqrt{5}\left(\sqrt{15}-\sqrt{5}\right)$
(o) $\sqrt{8}\left(\sqrt{32}-\sqrt{2}\right)$

6 Expand and simplify these expressions, leaving all surds in their simplest form.

(a) $\left(1+\sqrt{2}\right)\left(\sqrt{3}+4\right)$
(b) $\left(2+\sqrt{3}\right)\left(\sqrt{5}-1\right)$
(c) $\left(\sqrt{2}+\sqrt{3}\right)\left(\sqrt{2}+\sqrt{5}\right)$
(d) $\left(\sqrt{3}-\sqrt{2}\right)\left(2+\sqrt{3}\right)$
(e) $\left(\sqrt{3}-2\sqrt{2}\right)\left(\sqrt{5}+\sqrt{3}\right)$
(f) $\left(2+\sqrt{3}\right)\left(4-2\sqrt{3}\right)$
(g) $\left(3\sqrt{2}+\sqrt{5}\right)\left(\sqrt{2}+7\right)$
(h) $\left(2\sqrt{5}-\sqrt{3}\right)\left(\sqrt{3}+\sqrt{5}\right)$
(i) $\left(5\sqrt{2}-\sqrt{5}\right)\left(3\sqrt{2}+\sqrt{5}\right)$
(j) $\left(\sqrt{11}+2\sqrt{13}\right)\left(\sqrt{13}-2\sqrt{11}\right)$
(k) $\left(2\sqrt{5}+\sqrt{2}\right)\left(3\sqrt{2}+4\sqrt{5}\right)$
(l) $\left(4+2\sqrt{3}\right)\left(1-3\sqrt{3}\right)$
(m) $\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}+\sqrt{2}\right)$
(n) $\left(7+2\sqrt{5}\right)\left(2\sqrt{5}-7\right)$
(o) $\left(3\sqrt{3}-\sqrt{5}\right)\left(3\sqrt{3}+\sqrt{5}\right)$
(p) $\left(2\sqrt{2}+3\right)\left(2\sqrt{2}-3\right)$
(q) $\left(3\sqrt{2}+\sqrt{7}\right)\left(3\sqrt{2}-\sqrt{7}\right)$
(r) $\left(4\sqrt{5}-3\sqrt{6}\right)\left(4\sqrt{5}+3\sqrt{6}\right)$

7 Simplify $\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)$.

8 Expand and simplify these squares, leaving all surds in their simplest form:

(a) $\left(\sqrt{2}+3\right)^2$
(b) $\left(1-\sqrt{3}\right)^2$
(c) $\left(\sqrt{5}+4\right)^2$
(d) $\left(\sqrt{3}+\sqrt{2}\right)^2$
(e) $\left(\sqrt{10}-\sqrt{2}\right)^2$
(f) $\left(\sqrt{8}+\sqrt{2}\right)^2$
(g) $\left(\sqrt{6}-\sqrt{10}\right)^2$
(h) $\left(4\sqrt{2}-\sqrt{5}\right)^2$
(i) $\left(2\sqrt{2}+4\sqrt{5}\right)^2$
(j) $\left(3\sqrt{3}-2\sqrt{7}\right)^2$
(k) $\left(3\sqrt{2}+4\sqrt{6}\right)^2$
(l) $\left(4\sqrt{2}-5\sqrt{18}\right)^2$

9 Expand:
(a) $\left(\sqrt{a}+\sqrt{b}\right)^2$
(b) $\left(\sqrt{a}-\sqrt{b}\right)^2$

10 (a) Use a calculator to evaluate (to 4 sf):

(i) $\sqrt{5}+\sqrt{2}$
(ii) $\sqrt{5+2}$

(b) Show that $\sqrt{a}+\sqrt{b}\neq\sqrt{a+b}$.

11 (a) Use a calculator to evaluate (to 4 sf):
(i) $\sqrt{29}-\sqrt{17}$ (ii) $\sqrt{29-17}$

(b) Show that $\sqrt{a}-\sqrt{b}\neq\sqrt{a-b}$.

The darkroom and the digital clock

The only source of illumination in a photographer's darkroom is an alarm clock with a 'LED' (light-emitting diode) display in red.

The clock uses the 24-hour clock to display times. Each time consists of a number between 0 and 23, and then a colon, always followed by 2 digits.

- When is the room at its lightest?
- When is the room at its darkest?

Surd equations

A **surd equation** is one with a square root term in it. To solve a surd equation, square both sides to eliminate the surds. This can sometimes introduce an extra solution which does not fit the original equation, so each solution should be checked by substitution.

Example

Solve the equation $\sqrt{2x-1} = x - 2$.

Answer

$$\sqrt{2x-1} = x-2$$
$$2x-1 = (x-2)^2$$
$$2x-1 = x^2-4x+4$$
$$x^2-6x+5 = 0$$
$$(x-5)(x-1) = 0$$
$$x = 5 \text{ or } 1$$

Check each solution by substitution:

$\sqrt{2\times 5-1} = 5-2$?? Yes, $3 = 3$

$\sqrt{2\times 1-1} = 1-2$?? No, $\sqrt{1} \neq {}^{-}1$

Solution is $x = 5$.

EXERCISE 10.6

Solve these surd equations. Check all answers by substitution.

1 $\sqrt{3x} = \sqrt{x+6}$

2 $\sqrt{5x-2} = \sqrt{x+3}$

3 $x-2 = \sqrt{x}$

4 $x+1 = \sqrt{x+3}$

5 $\sqrt{11x+5} = x+3$

6 $x+2 = \sqrt{5x+6}$

7 $2x-1 = 3\sqrt{x-1}$

8 $\sqrt{4-x} = \sqrt{5}(2+x)$

9 $3\sqrt{4+x} = {}^{-}2\sqrt{x-1}$

10 $2\sqrt{3-x} = 5\sqrt{x+4}$

11 $3\sqrt{x} = 2\sqrt{1-4x}$

12 $2\sqrt{x+7} = x-1$

13 $\sqrt{x+10} = \sqrt{2}(x-5)$

14 $\sqrt{x+1} + \sqrt{2x} = 7$

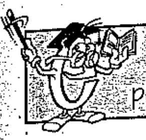
15 The sum of a number and its square root is 25 440. What is the number?

11 Logarithms

What are logarithms? **Logarithms** are indices of special base numbers. These **base numbers** are usually either:

- 10, which gives so-called **common logarithms**, or
- e, a special number in Mathematics, which gives so-called **natural logarithms** ($e \approx 2.72$).

In Year 12 we work with common logarithms only, and leave natural logarithms to Year 13.



Note that the base must be a positive number, and not equal to 1.

Calculators can work out logarithms. The abbreviation for logarithm is **log**. Log can also stand for common log (to base 10).

We say that $\log(2) = 0.3010$, because $10^{0.3010} = 2$. 0.3010 is the power of 10 needed to give 2.

In exactly the same way, $\log(1000) = 3$, because $10^3 = 1000$.

Example

Calculate log(5).

Answer

$\log(5) = 0.6990$ (on calculator)

Historical background

Imagine a world without readily available and affordable electronic calculators and computers. This was Planet Earth within the memory of most adults!

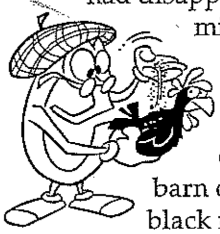
In such a world, multiplication and division of numbers was difficult and time-consuming. Although you may have been taught how to do long multiplication and division, the chances are that you have not had recent practice at it. Unless you are particularly expert and careful, you would find it difficult to carry out without making mistakes.

Logarithms were invented by John Napier (1550–1617), a Scottish mathematician. He is also famous for inventing the decimal point. Without the work pioneered by Napier, it would have been very difficult for subsequent mathematicians like Newton to have made their discoveries.

Napier's work on logarithms was somewhat of a hobby, and he is famous for much more than his mathematical discoveries. To defend Scotland against King Philip of Spain he suggested the concepts of the submarine, the tank and machine gun. Napier was also a keen farmer, and particularly interested in increasing production by spreading fertiliser on fields.

John Napier was reputed to be a magician, and stories are told of his supernatural activities in a small room at the top of his home, called Merchiston Tower, in Edinburgh. Folklore says that he carried a spider in a box, and had a black rooster as a spiritual familiar. This rooster features in a famous story told about Napier.

One day Napier realised that some tools had disappeared. They were going missing little by little. So, one day he spread some soot on the back of the black rooster, and then ordered all of his employees to go inside his barn one by one and pet the magic black rooster. He told them it would be able to tell who was stealing the tools. When everyone came out he told them to hold their palms up, and all but one had black palms. Napier knew that one was the thief because he was afraid to pet the rooster.



Another story relates how John Napier was annoyed by his neighbour's pigeons pecking in his field. The neighbour told John that if he could catch the pigeons he could have them.

So Napier soaked some peas in brandy, and spread them in his field for the pigeons to eat. The pigeons ate them, got drunk, and John was able to gather them up in his hands and put them in bags.

Here is what Napier wrote in 1614 about his discovery of logarithms. This was published in a Latin text called *Mirifici logarithmorum canonis descriptio*, and has been translated into English.

> *Seeing there is nothing (right well-beloved Students of the Mathematics) that is so troublesome to mathematical practice, nor that doth more molest and hinder calculators, than the multiplications, divisions, square and cubical extractions of great numbers, which besides the tedious expense of time are for the most part subject to many slippery errors, I began therefore to consider in my mind by what certain and ready art I might remove those hindrances. And having thought upon many things to this purpose, I found at length some excellent brief rules to be treated of (perhaps) hereafter. But amongst all, none more profitable than this which together with the hard and tedious multiplications, divisions, and extractions of roots, doth also cast away from the work itself even the very numbers themselves that are to be multiplied, divided and resolved into roots, and putteth other numbers in their place which perform as much as they can do, only by addition and subtraction, division by two or division by three.*

Logarithms have been used for hundreds of years in multiplication and division problems. Their great advantage is that they turn a multiplication problem into an addition problem, and a division problem into a subtraction problem. Below is an example of how logarithms were used.

Example

Here is a very simple multiplication problem:

$2 \times 3.$

2, as a power of 10, is $10^{0.3010}$—i.e. $\log(2) = 0.3010$.

3, as a power of 10, is $10^{0.4771}$—i.e. $\log(3) = 0.4771$.

This multiplication problem could be rewritten as

$10^{0.3010} \times 10^{0.4771}$

$= 10^{0.7781}$ (adding indices)

$= 5.999$ (4 sf) (using $\boxed{10^x}$ key)

Mathematicians had access to special tables of logarithms for these purposes. They also needed tables called antilogarithms to convert back from logarithms to ordinary numbers. Nowadays, we use calculators.

Formal definition of logarithms

If $b^p = q$,	then $\log_b(q) = p$
index form	log form

b is called the **base**, p is called the **logarithm**, and q is the **number**.

We say $\log_b(q)$ as 'log of q to base b'.

Example

Write a log statement equivalent to $3^4 = 81$.

Answer

$3^4 = 81$ is the same as $\log_3(81) = 4$.

Example

Write the statement $\log_2(128) = 7$ in index form.

Answer

$\log_2(128) = 7$ is the same as $2^7 = 128$.

In some cases we can use our knowledge of powers of numbers to find logarithms of whole-number bases and to solve simple equations.

Example

Work out (a) $\log_5(125)$

(b) $\log_{10}(100)$.

Answer

(a) 5 to what power gives 125? $5^3 = 125$.
Therefore $\log_5(125) = 3$.

(b) What power of 10 gives 100? $10^2 = 100$.
Therefore $\log_{10}(100) = 2$.

Example

Solve the equation $\log_3(x) = 2$.

Answer

This equation is equivalent to $3^2 = x$.
i.e. $x = 3^2 = 9$

Example

Find the value of p when $\log_p(64) = 6$.

Answer

$\log_p(64) = 6$ is a solution of the equation $p^6 = 64$.
To solve this equation take the 'sixth root' $\left(\text{i.e. power of } \frac{1}{6}\right)$ of both sides.

$p = \sqrt[6]{64} = 2$

Note: $(^-2)^6 = 64$, but p must be positive (bases of logarithms are positive numbers) so we take $p = 2$, not $^-2$.

EXERCISE 11.1 **Conversions between $b^p = q$ and $\log_b(q) = p$**

1 Write a log statement equivalent to each of these index statements:

(a) $36 = 6^2$
(b) $125 = 5^3$
(c) $7^3 = 343$
(d) $(1.2)^5 = 2.488\ 32$
(e) $6^{-1} = \frac{1}{6}$
(f) $\frac{1}{27} = 3^{-3}$
(g) $0.001 = 10^{-3}$
(h) $5^{-3} = 0.008$
(i) $a^b = c$
(j) $4^{-x} = y$

2 Write the following statements in index form:

(a) $3 = \log_5(125)$
(b) $\log_7(2401) = 4$
(c) $\log_{0.2}(0.000\ 064) = 6$
(d) $5 = \log_2(32)$
(e) $\log_4(64) = 3$
(f) $\log_2(256) = 8$
(g) $\log_b(a) = c$
(h) $q = \log_r(p)$
(i) $8 = \log_{\sqrt{2}}(16)$
(j) $\log_{\sqrt{5}}(25) = 4$

3 Find the value of q in each log statement:

(a) $\log_5(q) = 2$
(b) $\log_3(q) = 4$
(c) $\log_4(q) = 3$
(d) $\log_{10}(q) = 2$
(e) $\log_2(q) = 5$
(f) $\log_4(q) = 4$
(g) $\log_7(q) = 1$
(h) $\log_3(q) = 0$

4 Solve these equations:

(a) $\log_2(x) = 5$
(b) $\log_5(x) = 2$
(c) $\log_{16}(x) = \frac{3}{2}$
(d) $\log_{64}(x) = \frac{1}{2}$
(e) $\log_{\sqrt{2}}(x) = 8$
(f) $\log_{\frac{1}{3}}(x) = {}^-2$
(g) $\log_{\frac{1}{4}}(x) = {}^-4$
(h) $\log_8(x) = \frac{^-2}{3}$

5 Find the values of p in each log statement:

(a) $\log_p(64) = 3$
(b) $\log_p(25) = 2$
(c) $\log_p(9) = 2$
(d) $\log_p(32) = 5$
(e) $\log_p(1000) = 3$
(f) $\log_p(16) = 4$
(g) $\log_p(64) = 6$
(h) $\log_p(1) = 0$

6 Solve these equations:

(a) $\log_p(256) = 4$
(b) $\log_p(125) = 3$
(c) $\log_p(49) = 2$
(d) $\log_p\left(\frac{1}{81}\right) = 4$
(e) $\log_p(16) = \frac{4}{3}$
(f) $\log_p(27) = \frac{3}{4}$
(g) $\log_p\left(\frac{1}{25}\right) = {}^-2$
(h) $\log_p\left(\frac{1}{27}\right) = \frac{^-3}{2}$

7 Write down the value of these logarithms:

(a) $\log_2(8)$
(b) $\log_3(81)$
(c) $\log_5(25)$
(d) $\log_3(9)$
(e) $\log_4(64)$
(f) $\log_2(32)$

8 Find the value of x in each of these equations:

(a) $\log_3(4x) = 5$
(b) $\log_{4x}(64) = 2$

9 Use the properties of surds and indices to evaluate these logarithms:

(a) $\log_{\sqrt{3}}(27)$
(b) $\log_{\frac{1}{3}}(243)$
(c) $\log_{\sqrt{2}}(32)$
(d) $\log_{\frac{1}{9}}(27)$
(e) $\log_{\frac{1}{4}}(256)$
(f) $\log_3\left(\frac{1}{27}\right)$
(g) $\log_{\sqrt{5}}\left(\frac{1}{125}\right)$
(h) $\log_{10}(0.0001)$

10 Solve these equations:

(a) $\log_2(x) = {}^-3$
(b) $\log_{10}(x) = 0$
(c) $\log_2(x+1)^3 = 3$

Properties of logarithms

Logarithms obey three rules:

1 $\log(ab) = \log(a) + \log(b)$
When multiplying numbers, *add* their logarithms.

2 $\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$
When dividing numbers, *subtract* their logarithms.

3 $\log(a^n) = n\log(a)$
When raising a number to a power, *multiply* the logarithm by that power.

Because these rules apply to logarithms of all bases, there is no need to write in a base.

Example

Simplify $\log(3) + \log(5)$.

Answer

From rule 1 above,
$\log(3) + \log(5) = \log(3 \times 5) = \log(15)$

Example

Simplify $\log(12) - \log(2)$.

Answer

From rule 2 above,
$\log(12) - \log(2) = \log\left(\frac{12}{2}\right) = \log(6)$

Example

Write $3\log(2)$ as the log of a single number.

Answer

$3\log(2) = \log(2^3) = \log(8)$

Harder examples involve using more than one property at a time.

Example

Simplify $4\log(2) + \log(3) - \log(48)$.

Answer

$$\begin{aligned} 4\log(2) + \log(3) - \log(48) &= \log(2^4) + \log(3) - \log(48) \\ &= \log(16) + \log(3) - \log(48) \\ &= \log(48) - \log(48) \\ &= 0 \end{aligned}$$

Example

Simplify $\frac{\log(64)}{\log(4)}$.

Answer

$$\begin{aligned} \frac{\log(64)}{\log(4)} &= \frac{\log(4^3)}{\log(4)} \\ &= \frac{3\log(4)}{\log(4)} \\ &= 3 \end{aligned}$$

EXERCISE 11.2

1 Write as the log of a single number:

(a) $\log(2) + \log(3)$
(b) $\log(6) + \log(8)$
(c) $\log(4) + \log(5)$
(d) $\log(p) + \log(q)$
(e) $\log(x) + \log(y)$
(f) $\log(2) + \log(5) + \log(3)$
(g) $\log(4) + \log(8) + \log(2)$
(h) $\log(6) + \log(6)$

2 Write as the log of a single number:

(a) $\log(6) - \log(3)$
(b) $\log(18) - \log(2)$
(c) $\log(64) - \log(16)$
(d) $\log(14) - \log(7)$
(e) $\log(c) - \log(d)$
(f) $\log(8a) - \log(4a)$
(g) $\log(5) - \log(30)$
(h) $\log(7) - \log(63)$
(i) $\log(4) + \log(6) - \log(3)$
(j) $\log(2) - \log(12) + \log(6)$
(k) $\log(18) - \log(9) - \log(2)$
(l) $\log(54) - (\log(6) + \log(9))$

3 Write in terms of $\log(a)$, $\log(b)$ and $\log(c)$:

(a) $\log(abc)$
(b) $\log\left(\frac{ab}{c}\right)$
(c) $\log\left(\frac{b}{ac}\right)$
(d) $\log\left(\frac{1}{abc}\right)$

4 Write as the log of a single number:

(a) $2\log(5)$
(b) $6\log(2)$
(c) $3\log(5)$
(d) $2\log(10)$

5–7 *Simplify by writing as the log of a single number:*

5 (a) $3\log(4) - 2\log(2)$
(b) $5\log(2) + 2\log(6)$
(c) $2\log(3) - 3\log(2)$
(d) $3\log(5) - \log(10) + 3\log(2)$

6 (a) $\frac{1}{2}\log(16)$
(b) $\frac{1}{3}\log(27)$
(c) $\frac{3}{4}\log(81)$
(d) $\frac{1}{3}\log\left(\frac{1}{125}\right)$

add

7 (a) $\frac{1}{2}\log(25) + 2\log(10) - 3\log(5)$
(b) $\frac{1}{3}\log(64) - 5\log(2) + 3\log(4)$
(c) $\frac{2}{3}\log(27) + \frac{4}{3}\log(64)$
(d) $\frac{3}{5}\log(32) - \frac{3}{4}\log(16)$
(e) $\frac{5}{2}\log(4) - \frac{1}{2}\log(64)$
(f) $\frac{1}{4}\log(625) + 2\log(5) + \frac{1}{2}\log\left(\frac{1}{100}\right)$

8 Write in terms of $\log(p)$, $\log(q)$ and $\log(r)$:

(a) $\log(p^2)$
(b) $\log(pq^3)$
(c) $\log(\sqrt{pqr})$
(d) $\log(p^2q^3r^4)$
(e) $\log(r\sqrt{p})$
(f) $\log\left(p^2\sqrt[3]{q}\right)$
(g) $\log\left(\sqrt{p}\cdot\sqrt[4]{q}\cdot\sqrt[6]{r}\right)$
(h) $\log\left(\frac{pq}{\sqrt{r}}\right)$
(i) $\log\left(\frac{\sqrt{q}}{pr}\right)$
(j) $\log\left(\sqrt[4]{\frac{p^3}{q}}\right)$

9 Using the values $\log(2) = 0.301$ and $\log(5) = 0.699$, *hence* find:

(a) $\log(10)$
(b) $\log\left(\frac{5}{2}\right)$
(c) $\log(0.4)$
(d) $\log(4)$
(e) $\log(125)$
(f) $\log(20)$

10 If $\log(3) = p$, $\log(4) = q$ and $\log(11) = r$, find expressions for:

(a) $\log(12)$
(b) $\log(132)$
(c) $\log\left(\frac{11}{4}\right)$
(d) $\log(16)$
(e) $\log(99)$
(f) $\log(2)$

11 Simplify:

(a) $\frac{\log(36)}{\log(6)}$
(b) $\frac{\log(32)}{\log(2)}$
(c) $\frac{\log(125)}{\log(25)}$

continues...

(d) $\dfrac{\log(7)}{\log(49)}$

(e) $\dfrac{\log(64)}{\log(32)}$

(f) $\dfrac{3\log(2) + \log(4)}{\log(8)}$

(g) $\dfrac{\log(5) + 2\log(4) - \log(10)}{2\log(4) + \log(2)}$

(h) $\dfrac{\log(3) + 2\log(10) + \frac{1}{2}\log\left(\frac{1}{144}\right)}{3\log(10) - 3\log(2)}$

12 Simplify the following log expressions. You may need to use the properties $\log_x(1) = 0$ and $\log_x(x) = 1$.

(a) $\log_5(5)$

(b) $2\log_6(6)$

(c) $\log_2(4) + \log_2(16)$

(d) $\frac{1}{3}\log_3(27) + \log_3(3)$

(e) $\log_4(64) + \log_4(16) - \log_4(4096)$

(f) $\log_7(117\,649) - \log_7(40\,353\,607) + \log_7(2401)$

(g) $2\log_2(12) - \frac{1}{2}\log_2(16) - \log_2(36)$

(h) $\log_3(27) - \log_3(243)$

(i) $\log_2(1024) - \log_2(32\,768)$

(j) $\dfrac{\frac{1}{2}\log_{10}(100) - \log_{10}(10)}{\log_2(4) + \log_2(32)}$

13 Solve this equation for x:

$\log(4) + \log(x) = \log(8)$

14 Solve these equations:

(a) $\log(x) - \log(3) = \log(6)$

(b) $2\log(x) - \log(4) = \log(9)$

(c) $\log(x - 4) = \log(x) - \log(4)$

Index equations

Logarithms can be used to solve 'index' equations; these equations have x in the exponent or index. The approach is to take logarithms of both sides.

Example

Solve the equation $2^x = 17$.

Answer

$$2^x = 17$$
$$\log(2^x) = \log(17)$$
$$x\log(2) = \log(17)$$
$$x = \frac{\log(17)}{\log(2)} = \frac{1.2304}{0.3010} = 4.087$$

Note that the 4.087 above was calculated using the full calculator values for log(17) and log(2). But if 1.2304 is divided by 0.3010, then the result is 4.088. This shows that it is best not to round an answer until the *end* of a problem.

Example

Evaluate $\log_4(32)$.

Answer

Let $x = \log_4(32)$.

This is equivalent to solving the equation $4^x = 32$.

$$4^x = 32$$
$$\log(4^x) = \log(32) \text{ (taking logs of both sides)}$$
$$x\log(4) = \log(32)$$
$$x = \frac{\log(32)}{\log(4)} = \frac{\log(2^5)}{\log(2^2)} = \frac{5\log(2)}{2\log(2)} = \frac{5}{2}$$

Alternatively, $\log(32) \div \log(4) = 2.5$ on a calculator.

i.e., $\log_4(32) = \frac{5}{2}$

EXERCISE 11.3

1 Solve these equations. Give answers to 4 sf.

(a) $3^n = 12$
(b) $5^x = 3$
(c) $2^y = 6.4$
(d) $3^x = 2.3$
(e) $8^x = 92$
(f) $5^x = 28$
(g) $2^y = 10$
(h) $3^x = 18.5$
(i) $7^y = 24$
(j) $6^x = 256$

2 Evaluate these logarithms:

(a) $\log_2(8)$
(b) $\log_2(32)$
(c) $\log_5(125)$
(d) $\log_7(49)$
(e) $\log_{0.1}\left(\frac{1}{100}\right)$
(f) $\log_{\frac{1}{3}}(81)$
(g) $\log_{\frac{1}{2}}(16)$
(h) $\log_{\frac{1}{4}}(64)$
(i) $\log_{10}(0.1)$
(j) $\log_7(1)$

3 Solve these index equations. Give answers to 4 sf.

(a) $2^{x+1} = 39$
(b) $3^{x-5} = 18$
(c) $7^{2x+3} = 60$
(d) $13^{4x-5} = 6$

4 Evaluate these logarithms:

(a) $\log_3(5)$
(b) $\log_2(6)$
(c) $\log_{0.1}(18)$
(d) $\log_{29}(42)$

5 Solve the equation $5^{3x} = 0.04$.

6 Solve these index equations. Give answers to 4 sf.

(a) $4^x \times 5^x = 60$ $= (4\times5)^x = 60$. log(20
(b) $2^{3x} \times 4^{5x} = 120$
(c) $2^{x+1} \times 3^x = 68$

12 Sequences

A **sequence** is defined to be a mapping from the natural numbers to the real numbers. What does this mean?

A mapping is a formula, e.g.:

$$\left\langle \frac{6}{n} + 3 \right\rangle$$

into which we substitute the natural numbers $\{1, 2, 3, 4, \ldots\}$ to get another set of numbers, the sequence.

We use the symbol t_n to represent the **general** term, or nth term, of a sequence. The formula for a sequence, or the terms of a sequence, are enclosed in diamond-shape brackets—i.e. < >.

Example

Write down the first six terms of the sequence given by $\left\langle \frac{6}{n} + 3 \right\rangle$.

Answer

$t_1 = \frac{6}{1} + 3 = 9$

$t_2 = \frac{6}{2} + 3 = 6$

$t_3 = \frac{6}{3} + 3 = 5$

$t_4 = \frac{6}{4} + 3 = 4.5$

$t_5 = \frac{6}{5} + 3 = 4.2$

$t_6 = \frac{6}{6} + 3 = 4$

The sequence is $\langle 9, 6, 5, 4.5, 4.2, 4, \ldots \rangle$

EXERCISE 12.1

1 Write down the next 3 terms in these sequences:

(a) $1, 3, 5, 7, \ldots$

(b) $4, 2, 1, \frac{1}{2}, \ldots$

(c) $2, {}^{-}6, 18, {}^{-}54, \ldots$

(d) $1, 4, 9, 16, \ldots$

(e) $2, 3, 5, 8, 12, \ldots$

(f) ${}^{-}15, {}^{-}11, {}^{-}7, {}^{-}3, \ldots$

(g) $1, 8, 27, 64, \ldots$

(h) $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

(i) $3, 7, 15, 27, \ldots$

2 Write down the first four terms of the sequences with these general terms:

(a) $3n + 2$

(b) $\frac{n+2}{n}$

(c) $\frac{n+3}{2n-1}$

(d) $n^2 + n$

(e) $({}^{-}1)^n.n$

(f) 2^n

(g) $\frac{1}{n}$

(h) $\frac{n^2-1}{n+1}$

(i) $\frac{n}{2-n}, \quad n > 2$

3 Write down the first three terms of the sequence given by $t_n = n + \frac{1}{n}$.

4 (Multichoice) Which sequence is formed from the formula $t_n = 3 + 2n^2$?

A $3, 5, 11, 21, \ldots$

B $3, 5, 7, 9, \ldots$

C $5, 11, 21, 35, \ldots$

D $5, 7, 9, 11, \ldots$

E $5, 20, 45, 80, \ldots$

5–7 *And now some problems for lateral thinkers. Write down the next two terms for each sequence.*

5 $2, 3, 5, 7, 11, \ldots$

6 $1, 1, 2, 3, 5, 8, \ldots$

7 $61, 52, 63, 94, 46, 18, \ldots$

8–13 *More problems for lateral thinkers. Find the pattern, and write down the next two letters or words.*

8 J, F, M, A, M, ...

9 O, T, T, F, F, S, S, ...

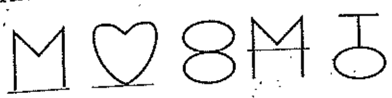
10 I, V, X, L, C, ...

11 choice, nature, degree, estate, column, ...

12 L, H, M, R, T, M, M, M, M, L, S, B, A, ...

13 H, M, K, R, M, L, P, M, B, ...

14 Draw the next diagram for this sequence:



(Hints for Questions 12 and 13 are at the end of the chapter.)

Recursive definition of sequences

Instead of having a formula that gives us each term of a sequence when natural numbers are substituted, an alternative approach is to have:

- the first term given to us, t_1
- a formula telling us how to get the *next term*, i.e. t_{n+1}, from the term before, t_n.

So we can get t_2 from t_1, and then get t_3 from t_2, and so on.

Such a definition is called **recursive**.

> It is important to understand how the labelling works here. The number in the subscript tells us which term it is in the sequence.
>
> t_1 = 1st term
>
> t_2 = 2nd term
>
> t_3 = 3rd term, etc.
>
> In general, then, t_n is the nth term. The term after the nth term is the $(n+1)$th term, i.e. t_{n+1}, and the term before the nth term is the $(n-1)$th term, i.e. t_{n-1}.
>
> The sequence is:
>
> $t_1, t_2, t_3, t_4, \ldots t_{n-1}, t_n, t_{n+1}, \ldots$

Example

The first term of a sequence is 3, and the $(n+1)$th term is given by $t_{n+1} = 5t_n - 11$. Write down the first four terms of this sequence.

Answer

If $n = 1$, $t_{n+1} = 5t_n - 11$ becomes
$t_2 = 5t_1 - 11 = 5 \times 3 - 11 = 4$

If $n = 2$, $t_{n+1} = 5t_n - 11$ becomes
$t_3 = 5t_2 - 11 = 5 \times 4 - 11 = 9$

If $n = 3$, $t_{n+1} = 5t_n - 11$ becomes
$t_4 = 5t_3 - 11 = 5 \times 9 - 11 = 34$

The sequence is $< 3, 4, 9, 34, \ldots >$

EXERCISE 12.2

1 Write down the first four terms of these recursively defined sequences:

(a) $t_{n+1} = t_n - 2$; $t_1 = 5$

(b) $t_{n+1} = 2t_n$; $t_1 = {}^-3$

(c) $t_{n+1} = 3t_n - 4$; $t_1 = 2$

(d) $t_{n+1} = 3t_n - 2$; $t_1 = {}^-1$

(e) $t_{n+1} = 1 - t_n$; $t_1 = {}^-6$

(f) $t_{n+1} = \dfrac{t_n}{4}$; $t_1 = 16$

(g) $t_{n+1} = (t_n)^2 - t_n$; $t_1 = {}^-2$

2 Calculate the second, third and fourth terms of these sequences:

(a) $t_{n+1} = 4t_n$; $t_1 = 4$

(b) $t_{n+1} = 3(t_n)^2$; $t_1 = 1$

3 Calculate the next four terms of these recursively defined sequences:

(a) $t_{n+1} = t_n - t_{n-1}$; $t_1 = 6, t_2 = 2$

(b) $t_{n+1} = 3t_n + 2t_{n-1}$; $t_1 = 1, t_2 = {}^-3$

(c) $t_{n+1} = \dfrac{t_n}{t_{n-1}}$; $t_1 = 6, t_2 = 3$

(d) $t_{n+2} = t_n - t_{n+1}$; $t_1 = {}^-3, t_2 = 4$

4 Write down the fourth term of the sequence defined by $t_n = t_{n-1} + n$; $t_1 = 5$.

5 Write down a recursive formula to identify each sequence below:

(a) $1, 3, 5, 7, \ldots.$

(b) $2, 4, 8, 16, \ldots.$

(c) $1, 3, 7, 15, 31, 63, \ldots.$

Sigma notation

The special symbol Σ is used to mean 'the sum of'. This symbol is the Greek capital letter *sigma*.

The numbers below and above the sigma symbol show the first and last terms to be added. These numbers are called **limits of summation.**

$$\sum_{\text{lower limit}}^{\text{upper limit}} (\text{each term}) = \sum_{\text{first term}}^{\text{last term}} (\text{each term})$$

Example

A sequence has $t_1 = 4,\ t_2 = 7,\ t_3 = 6,\ t_4 = 3,\ t_5 = 10 \ldots.$

Calculate:

(a) $\sum\limits_{i=1}^{4} t_i$ (b) $\sum\limits_{i=3}^{5} t_i$

Answer

(a) $t_1 + t_2 + t_3 + t_4 = 4 + 7 + 6 + 3 = 20$

(b) $t_3 + t_4 + t_5 = 6 + 3 + 10 = 19$

Sometimes the sequence formula (or rule) itself is placed after the sigma symbol. Then each individual term has to be calculated before adding.

Example

Evaluate $\sum\limits_{i=1}^{5} (4i - 2)$.

Answer

Substitute, in turn, each natural number between 1 and 5 into the formula $(4i - 2)$, and add.

$2 + 6 + 10 + 14 + 18 = 50$

When there is no possible doubt about the variable, it is sometimes omitted when writing down the limits of summation. Each of the expressions below is equivalent to the others:

$$\sum_{i=1}^{i=7} 5i = \sum_{i=1}^{7} 5i = \sum_{1}^{7} 5i$$

Each one represents the sum:

$5(1 + 2 + 3 + 4 + 5 + 6 + 7) = 5 \times 28 = 140$

Note also that the common factor of 5 meant that we did not have to multiply each number by 5 before adding.

EXERCISE 12.3

1 Given the sequence $\langle t_i \rangle = \langle 1, 4, 7, 10, 13, \ldots. \rangle$, evaluate:

(a) $\sum\limits_{1}^{3} t_i$ (b) $\sum\limits_{2}^{5} t_i$ (c) $\sum\limits_{1}^{6} t_i$

2 Write these sums in sigma notation:

(a) $t_1 + t_2 + t_3 + t_4 + t_5$

(b) $x_8 + x_9 + x_{10} + x_{11}$

3–15 Evaluate each of these sums:

3 $\sum\limits_{n=1}^{n=4} (2n - 3)$

4 $\sum\limits_{n={}^-2}^{n=2} n^2$

5 $\sum\limits_{x=1}^{x=5} (x + 4)(x - 3)$

6 $\sum\limits_{x={}^-3}^{1} (2x + 6)$

7 $\sum\limits_{x=2}^{6} (2x + 6)$

8 $\sum\limits_{2}^{5} \frac{n^2}{n - 1}$

9 $\sum\limits_{1}^{4} \frac{1}{n}$

10 $\sum\limits_{{}^-2}^{2} 2^n$

11 $\sum\limits_{2}^{5} \left(\frac{1}{2}\right)^n$

12 $\sum\limits_{{}^-3}^{1} ({}^-1)^i i^2$

13 Arithmetic sequences

In an **arithmetic sequence**, each term is calculated by adding or subtracting the same number each time.

This number is called the **common difference.**

Example

1, 7, 13, 19, 25, 31, ...	a sequence with a common difference of 6
5, 0, $^{-}5$, $^{-}10$, $^{-}15$, ...	a sequence with a common difference of $^{-}5$.

> We use the letter a to represent the **first term** of the arithmetic sequence.
>
> We use the letter d to represent the **common difference.**
>
> The formula for the **general term** (i.e. nth term) of an arithmetic sequence is:
>
> $t_n = a + (n-1)d$

Example

Calculate the 46th term of the arithmetic sequence 4, 9, 14, 19, ...

Answer

$a = 4 \qquad d = 5$

$t_{46} = 4 + (46-1) \times 5 = 4 + 45 \times 5 = 229$

Example

An arithmetic sequence has $t_4 = 46$ and $t_7 = 31$. Calculate the first term a and common difference d, and hence write down the first five terms of the sequence.

Answer

Use the formula for the general term:

$t_n = a + (n-1)d$

$t_4 = 46$	$46 = a + (4-1)d$	$46 = a + 3d$
$t_7 = 31$	$31 = a + (7-1)d$	$31 = a + 6d$

This gives us a pair of *simultaneous* equations in a and d:

$$\begin{aligned} a + 3d &= 46 \qquad (1) \\ a + 6d &= 31 \qquad (2) \\ ^{-}3d &= 15 \qquad (1)-(2) \\ d &= ^{-}5 \end{aligned}$$

Substituting $d = ^{-}5$ into (1) gives $a - 15 = 46$

$\Rightarrow a = 61.$

The sequence is < 61, 56, 51, 46, 41, ... >.

EXERCISE 13.1 **Finding terms in arithmetic sequences**

1 Write down the next two terms in these arithmetic sequences:

(a) 6, 11, 16, ...

(b) 30, 20, 10, ...

(c) $p, 3p, 5p, ...$

(d) $3c - 4b, c - 2b, ^{-}c, ...$

(e) $\frac{5}{8}, \frac{3}{8}, \frac{1}{8}, ...$

(f) 4.592, 3.205, 1.818, ...

2 Find the required term in the following sequences:

(a)	4, 6, 8, 10, ...	t_{50}
(b)	$^{-}5, ^{-}2, 1, 4, ...$	t_{26}
(c)	$7, 2, ^{-}3, ^{-}8, ...$	t_{40}
(d)	$16p, 14p, 12p, 10p, ...$	t_{35}
(e)	$^{-}1, 7, 15, 23, ...$	t_{32}
(f)	$3, 3\frac{1}{2}, 4, 4\frac{1}{2}, ...$	t_{27}
(g)	$6, 5\frac{3}{4}, 5\frac{1}{2}, 5\frac{1}{4}, ...$	t_{65}
(h)	13.198, 12.591, 11.984, 11.377, ...	t_{178}
(i)	$x, 2x + 2y, 3x + 4y, 4x + 6y, ...$	t_{15}

3 Find a formula for the nth term of these sequences:

(a) $1, 3, 5, 7, ...$

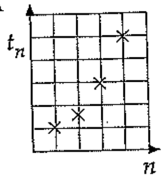
(b) $^-8, ^-3, 2, 7, ...$

(c) $\frac{1}{3}, \frac{4}{3}, \frac{7}{3}, \frac{10}{3}, ...$

(d) $1000, 900, 800, 700, ...$

4 (a) Which of these six graphs show terms from an arithmetic sequence?

A



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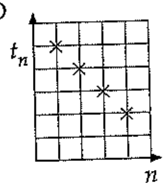

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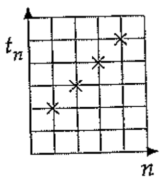

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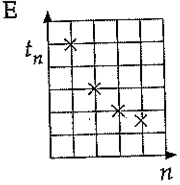

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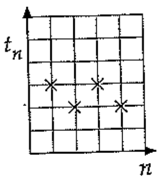

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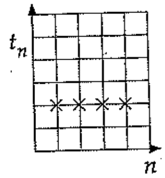


(b) Explain how you can tell if a graph shows an arithmetic sequence.

5 The first term of an arithmetic sequence is 8, and the common difference is $^-5$. What is the 22nd term?

6 How many terms for each of these sequences need to be taken until the given number is reached?

(a) $1, 3, 5, 7, ...$ 103

(b) $41, 36, 31, 26, ...$ $^-274$

(c) $1\frac{3}{4}, 2\frac{1}{2}, 3\frac{1}{4}, 4, ...$ $178\frac{3}{4}$

7 An arithmetic sequence has a first term of 15, and the 8th term is 43. What are the first four terms of the sequence?

8 How many terms of the sequence $< 2, 6, 10, 14, 18, ... >$ have a value of less than 297?

9 The first two terms of an arithmetic sequence are $a + 2b$ and $7b$. Find the 3rd term.

10 The 3rd term of an arithmetic sequence is 16, and the 9th term is 82. Find the first six terms of the sequence.

11 Calculate the first five terms of an arithmetic sequence which has a 12th term of 168 and a 5th term of 7.

12 Find the first term of an arithmetic sequence which has a 7th term of 167 and a 19th term of 23.

13 What is the common difference of the arithmetic sequence with 6th term of $^-56$ and 11th term of 11?

14 Find the 3rd term of the arithmetic sequence with $t_6 = 24$ and $t_{15} = 21$.

15 The first three terms of an arithmetic sequence are $< k, 2k + 5, 2 - k, ... >$. Calculate the value of k.

16 Find the first four terms of an arithmetic sequence where the sum of the 3rd and 4th terms is $^-7$, and the difference between the 14th and 19th terms is 15.

17 How many multiples of 4 are there between 156 and 196 inclusive?

18 How many multiples of 5 are there between 293 and 928?

19 (a) Calculate the number of digits needed to number the pages of these books:

(i) a book with 60 pages

(ii) a book with 600 pages

(iii) a book with 6000 pages.

(b) A book has n pages, where $100 \leq n \leq 1000$. Find a formula, in its simplest form, that gives the number of digits needed to number the pages of this book.

20 Show that $< \log a, \log ab, \log ab^2, \log ab^3, ... >$ is an arithmetic sequence. What is the common difference?

Arithmetic sequences: sum of first n terms

The formula for the sum of the first n terms of an arithmetic sequence is:

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Another formula that gives the sum is useful in some situations. It is:

$$S_n = \frac{n}{2}(a + l)$$

where a is the first term, and l is the last term.

Example

Calculate the sum of the first 20 terms of the sequence 2, 6, 10, 14, ...

Answer

$a = 2$ $\quad d = 4$ $\quad n = 20$

Substitute into $S_n = \frac{n}{2}[2a + (n-1)d]$.

$$\begin{aligned} S_{20} &= \frac{20}{2}[2 \times 2 + (20-1) \times 4] \\ &= 10(4 + 19 \times 4) \\ &= 10 \times 80 \\ &= 800 \end{aligned}$$

Example

A sequence is defined by $T(i) = 3i - 2$. Show that the sum of the first n terms of this sequence is given by $\frac{3n^2 - n}{2}$.

Answer

First write out some terms to get a feeling for the sequence:

$< 3i - 2 >$ gives the sequence $< 1, 4, 7, 10, 13, ... >$

Here $a = 1$ and $d = 3$:

$$\begin{aligned} S_n &= \frac{n}{2}[2a + (n-1)d] \\ &= \frac{n}{2}[2 \times 1 + (n-1) \times 3] \\ &= \frac{n}{2}(2 + 3n - 3) \\ &= \frac{n}{2}(3n - 1) \\ &= \frac{3n^2 - n}{2} \end{aligned}$$

EXERCISE 13.2 Summing terms in arithmetic sequences

1 Sum these arithmetic sequences to the number of terms indicated:

(a) $1, 4, 7, 10, ...$ — 15 terms

(b) $^-5, ^-3, ^-1, 1, ...$ — 28 terms

(c) $4, 5\frac{1}{2}, 7, 8\frac{1}{2}, ...$ — 18 terms

(d) $4p, 9p, 14p, 19p, ...$ — 61 terms

(e) $3, 3\frac{1}{3}, 3\frac{2}{3}, ...$ — 15 terms

(f) $8x, 5x, 2x, ^-x, ...$ — 16 terms

(g) $1.414, 1.97, 2.526, 3.082, ...$ — 37 terms

(h) $15x - 3y, 20x - y, 25x + y, 30x + 3y, ...$ — 666 terms

2 A sequence is defined by $T(i) = 5i + 1$. Find a formula for the sum of the first n terms of this sequence.

3 Find a formula for the sum of the first n terms of these sequences:

(a) $1, 3, 5, 7, ...$

(b) $^-8, ^-3, 2, 7, ...$

(c) $\frac{1}{3}, \frac{4}{3}, \frac{7}{3}, \frac{10}{3}, ...$

(d) $1000, 900, 800, 700, ...$

4 The sum of n terms of an arithmetic sequence with first term 4 and common difference 6 is 784. Calculate the value of n.

5 The first term of an arithmetic sequence is 8, and the sum of the first 20 terms is 730. Find the common difference.

6 The sum of the first 86 terms of an arithmetic sequence is 1290. If the common difference is 14, find the first term of the sequence.

7 An arithmetic sequence has a common difference of $^{-}5$, and the sum of the first 26 terms is 468. Calculate the first term of the sequence.

8 An arithmetic sequence has 3rd term 5, and common difference 2. Calculate:
(a) the first term
(b) the sum of the first 10 terms.

9 Calculate the first four terms of an arithmetic sequence, given that the sum of the first six terms is 15, and the sum of the first nine terms is 63.

10 The fifth term of an arithmetic sequence is 13, and the sum of the first five terms is 45. Calculate the first term and the common difference of the sequence.

11 Sum the multiples of 7 between 63 and 511 inclusive.

12 A special type of number is formed by arranging digits in order from smallest to largest so that there are the same number of digits as the 'value' of that digit:

1
122
122333
1223334444
122333444455555

(a) How many digits are needed if the last one is 5?
(b) How many digits are needed if the last one is 8?
(c) Write down a rule that gives the number of digits if the last one is n.

13 The sum of the first 20 terms of an arithmetic sequence is p, and the sum of the first 22 terms is q. Write down expressions for the common difference, and the sum of the first 18 terms.

13

INVESTIGATION – The parking building

Some architects are designing a large, multi-level parking building. The fixed cost for designing and managing the whole project is $1 200 000. The first level costs $3 500 000 to build, and then the cost of building each level is $75 000 more than the previous level.

First level	$3 500 000
Second level	$3 575 000
Third level	$3 650 000
....	...

The income per month for each level is $45 000.

The owners have to make a decision about how many levels to build to give the *maximum* percentage return per month.

For example, the percentage return for 2 levels is calculated as follows:

Income = 2 × $45 000 = $90 000

Cost of building: $1 200 000 + $3 500 000 + $3 575 000 = $8 275 000

$$\text{Percentage return} = \frac{90\,000}{8\,275\,000} \times \frac{100}{1} = 1.0876\%$$

Produce a spreadsheet that shows the percentage return for up to 11 levels.

This extract shows how the first few rows could appear:

	A	B	C	D	E
1	Number of levels	Cost of building	Total cost	Return	Percentage rate of return
2					
3	Fixed cost	1200000	1200000		
4	1	3500000	4700000	45000	0.9574
5	2	3575000	8275000	90000	1.0876
6	3	3650000	11925000	135000	1.1321
7	4				
8	5				
9	6				
10	7				
11	8				
12	9				
13	10				
14	11				

2 How many levels should be built so that the owners get the maximum possible return per month?

EXERCISE 13.3 Applications

1 Kim buys a car with an odometer reading of 55 600 km. What would be the reading after 40 days if Kim drives 90 km per day on average?

2 Paving stones are laid in a series of straight lines to cover a barbecue patio. The first line has 10 paving stones, and each successive row has one more stone than the previous row. The area is 66 rows deep. Calculate how many stones are needed to pave the patio.

3 When taking up jogging to keep fit, Bruce runs 500 m the first morning and adds a further 50 m each morning he goes out running. If Bruce runs four mornings a week, calculate:

(a) how far he will run on the final morning of the 5th week

(b) how far he will have run in total after 5 weeks of training.

4 A 'Dutch' auction is one in which the price drops by the same amount at each 'bidding opportunity' until someone is prepared to bid and pay the price.

A charity is given an air-ticket to Sydney and decides to sell it at a fund-raising dinner using a 'Dutch' auction. The price starts at $680 and drops by $15 each minute until someone buys it.

(a) If the successful buyer waited 18 minutes to bid, calculate how much they paid.

(b) Another buyer was prepared to pay $350. How many minutes would they wait to bid if they were the successful bidder?

5 In a Douglas fir nursery, trees are planted in rows—each row containing three less trees than the previous one. If the final row has 10 trees and the first row has 223, calculate:

(a) the number of rows

(b) the total number of trees.

6 The interest earned on an account in the first month is 10 cents, the second month 12 cents, and the third month 14 cents, and so on. Calculate the total interest earned after 191 months.

7 A transport planner has collected data about the number of passengers on a particular train journey at each stop from the first station to the centre of town. An arithmetic sequence makes a good model for the number of people on the train. Some of the data is given in this table:

	First station	Second station	Third station	....	Centre of town
Number of passengers	205	245	285		725

(a) The same number of passengers board the train at each station. What is this number?

(b) Write down the values of a and d in this arithmetic sequence.

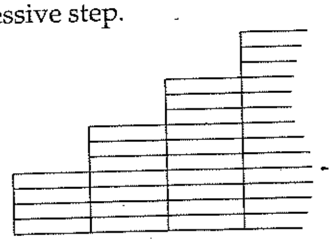
(c) Write down a formula for the number of passengers on the train at the nth station.

(d) Use the formula in (c) to write down and solve an equation for the number of stations on this journey.

8 A university student takes out a student loan for \$6500. The student spends \$150 per week on expenses. How many weeks will it take until the student only has \$2000?

9 A suit is priced at \$775 before being put on sale. The retailer realises that the suit is over-priced, and decides to reduce the price by \$25 each day in an attempt to sell it. How many days would it be before the suit could be bought for \$25?

10 A set of steps is built with paving stones as shown. The bottom step has 4 paving stones, and 3 paving stones are added for each successive step.



(a) How many paving stones will be needed for the 15th step?

(b) How many paving stones will have been used for the first 15 steps altogether?

11 A fish farmer starts with 400 salmon and sells 10 at the end of each week to a restaurant.

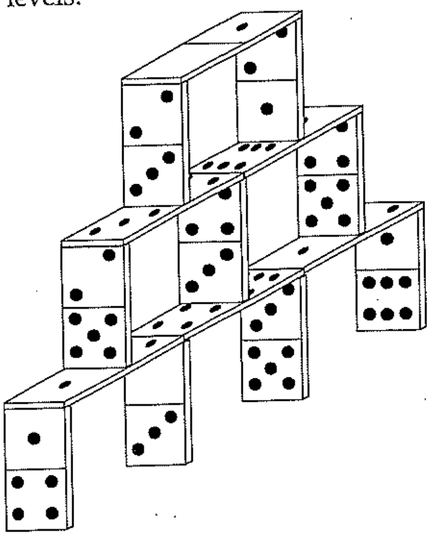
- Each salmon costs 50 cents a week to feed
- Each salmon sells for \$10.

(a) After how many weeks have all the salmon been sold?

(b) Calculate the total cost of feeding the salmon.

(c) How much money does the fish farmer make or lose altogether?

12 The diagram below shows a set of 'domino' steps. These can be built so that they have many levels.



(a) Copy and complete this table to show the number of dominoes needed altogether for different numbers of levels.

Number of levels	Number of dominoes needed to build the steps
1	3
2	8
3	
4	
5	

(b) Find an expression for the number of dominoes needed altogether when there are n levels. (*Hint:* use the formula for the sum of terms of an arithmetic sequence.)

13 An engineering graduate's starting salary was $24 750 per year. If she received annual increases of $5925, what would be:

(a) her salary after 7 years?

(b) her total earnings in that time?

14 Each step at the Chichen Itza pyramid in the Yucatan Peninsula of Mexico is 30 cm in height. The first step is 30 cm off the ground, the second is 60 cm off the ground, and so on. If all the heights are added together, the total of the heights of each step is 1864.8 m.

(a) How many steps are there on the side of the pyramid?

(b) What is the height of the top step?

15 A fencing contractor is building a rabbit fence in a straight line across a plain. He starts by using a truck to collect fence posts from a conveniently sited depot, and drops off 240 fence posts every 500 m along the intended route. After each delivery the truck returns to the depot to collect more posts.

(a) How long is the fence if a total of 4320 posts are delivered?

(b) What is the total distance travelled by the truck to deliver the 4320 posts?

16 The bars on a xylophone are cut so that each one is 4 mm longer than the one before. The shortest one is 129 mm. The sum of the lengths of all the bars is 4425 mm. Calculate:

(a) the number of bars

(b) the length of the longest bar.

17 A long stainless-steel tube, 19.5 m in length, is to be cut into six pieces to make a section of a pipe organ. Each piece is 700 mm longer than the previous one. Calculate the length of:

(a) the shortest piece

(b) the longest piece.

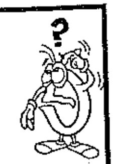
18 Billy and Howard are fundraising by participating in a sponsored walk from Picton to Bluff—total distance 949 km. Billy starts by walking north from Bluff, and Howard heads south from Picton. Billy walks 40 km the first day, 38 km the second day, 36 km the third day, and so on. Howard walks 12 km the first day, 13 km the second day, 14 km the third day, and so on.

(a) How far apart are they after one day?

(b) How far apart are they after two days?

(c) On what day do they meet?

The 51 pearls



51 pearls are threaded together on a string. The smallest (and cheapest) pearls are at each end. Starting from these, each pearl is worth $100 more than the one on its outside. The middle pearl is the largest and most expensive. Altogether the string of pearls is worth $139 000. Calculate the value of the middle pearl.

14 Geometric sequences

In a **geometric sequence**, each term is calculated by *multiplying* the previous term by the same number each time. This number is called the **common ratio**.

Examples

2, 8, 32, 128, ...	a sequence with a common ratio of 4
$36,\ ^{-}12,\ 4,\ \frac{^{-}4}{3},\ \ldots$	a sequence with a common ratio of $\frac{^{-}1}{3}$.

4

> We use the letter a to represent the **first term** of the geometric sequence, and the letter r to represent the **common ratio.**
>
> The formula for the **general term** (i.e. nth term) of a geometric sequence is:
>
> $t_n = a \times r^{n-1}$

Example

What is the value of r (the common ratio) in the geometric sequence 24, 36, 54, 81, ...?

Answer

To calculate r, divide any term by the previous term:

$$\frac{t_2}{t_1} = \frac{36}{24} = 1.5$$

As a check, divide two other consecutive terms. This should give the same result:

$$\frac{t_4}{t_3} = \frac{81}{54} = 1.5$$

Example

The first four terms of a geometric sequence are 6, 12, 24, 48, ...

(a) Calculate the 7th term.

(b) Write down a formula for the nth term.

Answer

(a) $a = 6 \qquad r = 2$

$$\begin{aligned} t_7 &= 6 \times 2^{7-1} \\ &= 6 \times 2^6 \\ &= 6 \times 64 \\ &= 384 \end{aligned}$$

(b) $$\begin{aligned} t_n &= a \times r^{n-1} \\ &= 6 \times 2^{n-1} \end{aligned}$$

Example

A geometric sequence has a 3rd term of 36, and a 5th term of 324. Calculate the common ratio, and the first term.

Answer

$t_3 = 36$	$36 = ar^{3-1}$	$\Rightarrow$	$36 = ar^2$	(1)
$t_5 = 324$	$324 = ar^{5-1}$	$\Rightarrow$	$324 = ar^4$	(2)

These are two simultaneous equations, which we solve by eliminating a first to find r. However, we do this not by subtracting, but by dividing (2) by (1).

$$\begin{aligned} \frac{ar^4}{ar^2} &= \frac{324}{36} \\ r^2 &= 9 \\ r &= \pm 3 \end{aligned}$$

Note that there are two possible values for r in this example.

To find a, substitute $r = \pm 3$ into equation (1):

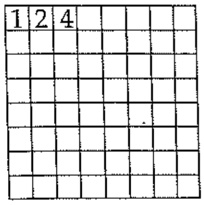
$$\begin{aligned} a \times 3^2 &= 36 \\ 9a &= 36 \\ a &= 4 \end{aligned}$$

SPREADSHEET INVESTIGATION – The grains of rice

The game of chess was invented many centuries ago, bably in China or ia. One legend sociated with its scovery goes like this:

The Emperor of China was so pleased with this new game that he offered the inventor any reward of his choice. The inventor thought for a while, and then said:

- give me one grain of rice for the first square of the chessboard on the first day
- then give me two grains for the second square on the next day
- then give me four grains for the third square on the next day, and so on.



The Emperor thought this was an insignificant reward, and promptly agreed.

A month (30 days) later, the Emperor, facing financial ruin, had the inventor's head cut off!

Facts about grains of rice:

- 1 grain of rice weighs about 2 mg (0.002 g)
- 1 tonne = 1000 kg = 1 000 000 g

1 Produce a spreadsheet that shows the number of grains of rice for each of the 64 squares, and the total number of grains given out at each stage.

This extract here shows how the first few rows could appear.

	A	B	C
1	Square number	Grains on that square	Total number of grains
2			
3	1	1	1
4	2	2	3
5	3	4	7
6	4	8	15
7	5	16	31

2 Produce a graph showing how the number of grains given out increases.

3 What is the total number of grains after 64 squares? Express your answer in standard form correct to 4 sf.

4 What is the weight of this total number (in tonnes)?

5 Use your knowledge of geometric sequences to write down a formula for the number of grains for the nth square.

6 Look carefully at the relationship between the numbers in columns B and C, and then suggest a formula for the *total* number of grains given out after the nth square is reached.

EXERCISE 14.1 Finding terms in geometric sequences

1 Write down the next three terms in the following geometric sequences:

(a) $6, 12, 24, 48, \ldots$

(b) $4, 2, 1, \frac{1}{2}, \ldots$

(c) $9, {}^{-}3, 1, \frac{{}^{-}1}{3}, \ldots$

(d) $2, 6, 18, 54, \ldots$

(e) $x, x^2, x^3, x^4, \ldots$

(f) ${}^{-}16, {}^{-}4, {}^{-}1, \frac{{}^{-}1}{4}, \ldots$

(g) $8, {}^{-}2, \frac{1}{2}, \frac{{}^{-}1}{8}, \ldots$

(h) $2x^3, {}^{-}4x^5, 8x^7, {}^{-}16x^9 \ldots$

(i) $1, {}^{-}1, 1, {}^{-}1, \ldots$

2 Write down the common ratio for each of the geometric sequences in Question 1 above.

3 Find the required term in the following geometric sequences. Give all answers as whole numbers or fractions.

(a) $5, 10, 20, 40, \ldots$ t_9

(b) $4, 2, 1, \frac{1}{2}, \ldots$ t_{10}

(c) $1, 3, 9, 27, \ldots$ t_{13}

(d) $16, 4, 1, \frac{1}{4}, \ldots$ t_8

4 Find the required term in the following geometric sequences. Give all answers in standard form correct to 4 sf.

(a) $8, 64, 512, 4096, \ldots$ t_7

(b) $1, 1.1, 1.21, 1.331, \ldots$ t_{29}

(c) $\frac{1}{9}, \frac{{}^{-}1}{3}, 1, {}^{-}3, \ldots$ t_{12}

(d) $128, 32, 8, 2, \ldots$ t_{14}

5 Write down the required term for each of these geometric sequences:

(a) $1, {}^{-}1, 1, {}^{-}1, \ldots$ t_{37}

(b) ${}^{-}2, 2, {}^{-}2, 2, \ldots$ t_{44}

(c) $8, {}^{-}8, 8, {}^{-}8, \ldots$ t_{308}

6 Find a formula for the nth term of these geometric sequences:

(a) $3, 6, 12, 24, \ldots$

(b) $180, 60, 20, 6\frac{2}{3}, \ldots$

(c) $\frac{{}^{-}1}{8}, \frac{1}{4}, \frac{{}^{-}1}{2}, 1, \ldots$

(d) $5, {}^{-}10, 20, {}^{-}40, \ldots$

(e) $64, 96, 144, 216, \ldots$

7 Write down the fifth term in the geometric sequence

$2, 2\sqrt{2}, 4, 4\sqrt{2}, \ldots$

8 Calculate the first term of these geometric sequences:

(a) with common ratio 4, and 9th term 786 432

(b) with common ratio 3, and 11th term 944 784

(c) with common ratio $\frac{1}{2}$, and 6th term 120

(d) with common ratio ${}^{-}2$, and 8th term 1920.

9 The first term of a geometric sequence is 28, and the 3rd term is 7. Calculate:

(a) the common ratio

(b) the 8th term.

10 Find a formula for the nth term of the geometric sequence $< 54x^5, 18x^4, 6x^3, \ldots >$

11 The first term of a geometric sequence is 216, and the 8th term is $1\frac{11}{16}$. Calculate the common ratio.

12 A geometric sequence has a 6th term of 32, and the 10th term is 512. Calculate:

(a) the value of r, the common ratio

(b) the value of a, the first term

(c) the 15th term.

13 The 4th term of a geometric sequence is 3, and the 8th term is $\frac{1}{27}$. Find the 12th term.

14 Calculate the second term of a geometric sequence which has a 3rd term of ${}^{-}5$ and an 8th term of $\frac{1}{625}$.

Find the third term in a geometric sequence which has a first term of $^{-}81$, and a fifth term of $\frac{^{-}1}{81}$.

18, x, 32 are three consecutive terms in a geometric sequence. Calculate the value(s) of x.

The first three terms in a geometric sequence are $< x+2, x-2, x-4, \ldots >$. Calculate the value of x.

$a^2 - b^2$ and $a - b$ are the first two terms in a geometric sequence. Find the third term.

19 A dishonest bartender sells 'Tropical' fruit drink. This is made up of mixture of pure pineapple juice (80%) and water (20%). The drink is stored in a 40-litre container. Each time the level of drink has dropped by 1 litre the bartender adds 1 litre of *water* to the container to refill it.

How much pure pineapple juice will be in the container after it has been refilled 10 times? Give your answer to the nearest mL.

Sum of the first n terms of a geometric sequence

The formula for the sum of the first n terms of a geometric sequence is:

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

If r is less than 1, it is more convenient to use the form:

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

This avoids having to simplify negative terms in the fraction.

Example

Calculate the sum (to the nearest whole number) of the first 9 terms of the geometric sequence 125, 225, 405, 729, ...

Answer

Clearly $a = 125$.

To find r, divide any term by the previous term:

$r = \frac{405}{225} = 1.8$

Substitute into $S_n = \frac{a(r^n - 1)}{r - 1}$.

$$S_9 = \frac{125\left[(1.8)^9 - 1\right]}{1.8 - 1}$$

$$= \frac{125[198.3593 - 1]}{0.8}$$

$$= 30\,837$$

EXERCISE 14.2 Summing terms in geometric sequences

1 Sum these geometric sequences for the indicated number of terms:

(a) 5, 10, 20, 40, ... 10 terms

(b) 1, 3, 9, 27, ... 12 terms

(c) 4, 2, 1, $\frac{1}{2}$, ... 16 terms

(d) 64, 16, 4, 1, ... 8 terms

2 Calculate the sum of these geometric sequences with negative common ratios for the required number of terms:

(a) 1, $^{-}7$, 49, $^{-}343$, ... 9 terms

(b) 16, $^{-}4$, 1, $\frac{^{-}1}{4}$, ... 10 terms

(c) $\frac{^{-}1}{8}$, $\frac{1}{4}$, $\frac{^{-}1}{2}$, 1, ... 12 terms

(d) $^{-}81$, 27, $^{-}9$, 3, ... 9 terms

3 Find the sum of the first 55 terms of these geometric sequences:

(a) $^{-}1$, 1, $^{-}1$, 1, ...

(b) 7, $^{-}7$, 7, $^{-}7$, ...

4 Find the sum of the first 84 terms of these geometric sequences:

(a) $2, {}^{-}2, 2, {}^{-}2, \ldots$

(b) ${}^{-}100, 100, {}^{-}100, 100, \ldots$

5 The sum of the first five terms of a geometric sequence is 19.375. If the common ratio is 0.5, find the first term.

6 The sum of the first eight terms of a geometric sequence is 3280. If the common ratio is 3, calculate the third term.

7 The formula for the sum of the first n terms of a geometric sequence is $S_n = 5^n - 1$. Find the first four terms of the sequence.

8 Use the formula for the sum of terms in a geometric sequence to estimate the total number of your ancestors born last millennium (1000 AD to 2000 AD). Make these assumptions:

- there is a gap of about 25 years between each generation
- there has been no inbreeding.

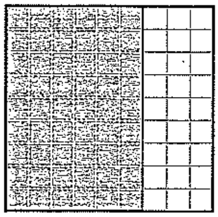
(a) Write your answer is standard form correct to 2 sf.

(b) The population of the world in 1000 AD was about 300 000 000. Which of the two assumptions is more likely to be wrong?

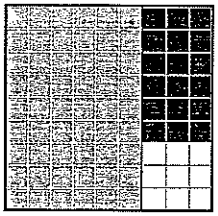
The sum to infinity of a geometric sequence

An architect is designing a pattern for a square stained-glass window. The window measures 81 m^2 altogether.

The pattern is made up as follows: the architect colours in $\frac{2}{3}$ of the window to begin with. Then another colour is chosen, and $\frac{2}{3}$ of the remaining part is coloured in. Then a third colour is chosen, and $\frac{2}{3}$ of the remaining part is coloured in, and so on.



54



54 + 18

54 + 18 + 6

The sequence of parts coloured in is:

54, 18, 6, ...

- What is the next number in this sequence?
- Explain why this sequence is geometric. What is the common ratio?
- Can this process continue indefinitely?

Another sequence produced by this process is 54, 72, 78, ...

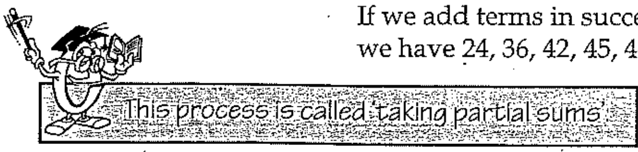
- Describe what the terms in this sequence represent.
- Explain whether the terms in this sequence will ever reach 80. Will they ever reach 81?

When a geometric sequence has a common ratio between ${}^{-}1$ and 1 (that is, ${}^{-}1 < r < 1$), then the terms become smaller in size as we move along the sequence.

Example

$\langle 24, 12, 6, 3, \ldots \rangle$

If we add terms in succession from this sequence we have 24, 36, 42, 45, 46.5, etc.



The terms in this sequence of 'partial sums' get closer and closer to 48 without ever reaching 48 exactly. We describe 48 as being the 'sum to infinity' of the sequence < 24, 12, 6, 3, ... >

Although a geometric sequence has an infinite number of terms, if the value of r is between $^-1$ and 1 we can give a meaning to what happens when we add them up term by term.

The total of all the terms is called the **sum to infinity**.

The formula for the sum to infinity of a geometric sequence is:

$S_\infty = \frac{a}{1-r}$ (for $^-1 < r < 1$ only)

Example

Find the sum to infinity for the sequence

$\left\langle 12, {}^-3, \frac{3}{4}, \frac{^-3}{16}, \ldots \right\rangle$.

Answer

$a = 12$ $\quad r = \frac{^-1}{4}$ (found by dividing $^-3$ by 12)

$$S_\infty = \frac{12}{1 - \frac{^-1}{4}}$$

$$= 12 \div \frac{5}{4}$$

$$= 9.6$$

Example

A sequence is < 96, 24, 6, ... >

(a) Explain why the sequence is a geometric sequence.

(b) Show why the sum of these terms cannot exceed 128.

Answer

(a) The sequence is geometric because each term is $\frac{1}{4}$ of the previous term. This means that each term is obtained by multiplying by a common ratio.

(b) $S_\infty = \frac{a}{1-r}$

$$= \frac{96}{1 - \frac{1}{4}}$$

$$= 96 \div \frac{3}{4}$$

$$= 96 \times \frac{4}{3}$$

$$= 128$$

EXERCISE 14.3

1 Calculate the sum to infinity of these geometric sequences:

(a) $4, 2, 1, \frac{1}{2}, \ldots$

(b) $81, 27, 9, 3, \ldots.$

(c) $9, 6, 4, \frac{8}{3}, \ldots$

(d) $75, 30, 12, \ldots.$

2 Calculate the sum to infinity of these geometric sequences with negative common ratios:

(a) $6, {}^-3, \frac{3}{2}, \frac{^-3}{4}, \ldots.$

(b) $12, -3, \frac{3}{4}, \frac{^-3}{16}, \ldots.$

(c) $^-64, 48, {}^-36, 27, \ldots.$

(d) $^-500, 100, {}^-20, 4, \ldots.$

3 Find the first term of these geometric sequences:

(a) with common ratio $\frac{3}{4}$ and sum to infinity of 24

(b) with common ratio $\frac{^-1}{6}$ and sum to infinity of $10\frac{2}{7}$.

4 Find the common ratio of these geometric sequences:

(a) with first term 15 and sum to infinity of $22\frac{1}{2}$

(b) with first term 24 and sum to infinity of 64

(c) with first term 84 and sum to infinity of 112.

5 (a) Find the sum to infinity of the geometric sequence $\left\langle \frac{7}{10}, \frac{7}{100}, \frac{7}{1000}, \ldots \right\rangle$.

(b) Write 0.7 + 0.07 + 0.007 + 0.0007 + ... as a recurring decimal.

(c) Hence write $0.\dot{7}$ as a fraction.

6 (a) Write 0.45 + 0.0045 + 0.000 045 + ... as a recurring decimal.

(b) Find the sum to infinity of the geometric sequence < 0.45, 0.0045, 0.000 045, ... >

(c) Hence express $0.\dot{4}\dot{5}$ as a fraction.

7 Use the method from Questions 6 and 7 to write these recurring decimals as fractions in their simplest form:

(a) $0.\dot{4}$ (c) $0.\dot{4}1\dot{5}$

(b) $0.\dot{1}\dot{8}$ (d) $0.4\dot{3}$

EXERCISE 14.4 Applications

1 A person who weighs 104 kg plans to lose 8 kg during the first three months of a diet, 6 kg during the next three months, 4.5 kg during the following three months, and so on. This pattern of weight loss forms a geometric sequence. Find the person's weight after a large number of years, assuming their diet is successful!

2 Kate rinses her hair after washing it with a shampoo. The first rinse removes 5 g of shampoo. Each successive rinse removes 20% of the amount removed by the previous rinse. How much shampoo was present in Kate's hair at the beginning of the wash?

3 The production from a market garden is declining as the soil fertility drops. The first crop of beans weighs 25 tonnes. Each year after that the total weight of beans produced falls by 10% (that is, it is 90% of the previous year's weight).

(a) What is the weight of beans produced in the 5th year?

(b) Estimate the total weight of beans that the market garden can produce over a long time.

4 A poplar tree sapling measures 1.2 m when first planted. After planting it grows 900 mm in the first year, and from then on each year's increase in height is $\frac{2}{3}$ of the previous year's increase. Calculate the maximum height (in metres) that the tree can reach.

5 A 100 litre container of toxic waste is buried in a landfill. Each year some of the contents leak out into the surrounding water-table. The amount leaking each year follows a geometric sequence:

first year:	15 litres
second year:	12 litres
third year:	9.6 litres

(a) Calculate the total amount that leaks over a very long period of time.

(b) How much toxic waste will remain in the container regardless of how long it is left buried?

6 The pendulum of a grandfather clock is running down. It usually swings through an angle of 10°, but now decreases by 1% of the previous angle with each swing.

(a) Calculate the total angle that the pendulum swings through before it comes to rest.

(b) If the length of the arm of the pendulum is 700 mm, calculate the total distance travelled by the pendulum. (*Hint*: use the arc length formula $s = r\theta$.)

7 A ball is dropped from a height of 5 m. With each bounce it rises $\frac{4}{5}$ of the height of the previous bounce. Calculate the total distance travelled by the ball before it comes to rest.

8 A square has sides of 16 cm. The middle points of its sides are joined to form a second square. The middle points of the sides of this square are joined to form a third square, and so on to form an infinite sequence of squares.

(a) Calculate the sum of the lengths of the sides of all the squares.

(b) Calculate the sum of their perimeters.

(c) Calculate the sum of the areas of all the squares.

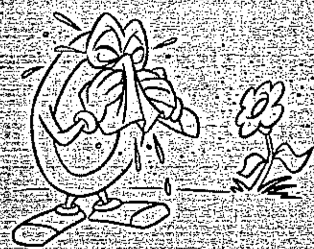
15 Growth and decay

...where quantities grow steadily or ...steadily can often be modelled or ...mated by terms of a geometric sequence. ...urs when each term in the sequence is multiplied by the same number to go to the ...term.

...**ematical model** is a ... graph, etc., that explains ...ximately in most cases) the ...nship between quantities in a real-life ...tion.

Example

...forced through a succession of filters to ...ve pollen spores. Each filter removes ...95% of the pollen spores. In springtime ...number of spores per m^3 in the air might ...about 500 000.



...is table shows the estimated number of spores per m^3 before the air is passed through the filters:

	1st filter	2nd filter	3rd filter	4th filter
Estimated number of spores	500 000	25 000	1250	?

The number of spores at each stage can be modelled by a geometric sequence formula. Here it would be $500\,000 \times (0.05)^{n-1}$.

There are many real-life situations in which a geometric sequence model can apply. This table gives *some* examples:

Growth	Decay
Population numbers	Radioactivity
Compound interest	Depreciation
Price inflation	Filtering/rinsing

We will look at some special cases below.

Compound interest

Compound interest is paid on money invested when interest is added to the **principal** (original amount invested) to form a new principal for the next term of a loan. In other words, interest is paid on interest.

Example

\$2000 is deposited into a bank account that pays 12% compound interest. The interest is added to the account once a year. Calculate the amount in the account after two years.

Answer

Interest for the first year = 12% × \$2000 = \$240
This is added to the principal at the end of the first year, to make a new principal of \$2240.

Interest for the second year = 12% × \$2240 = \$268.80
This is added to the principal at the end of the second year, to make a new principal of \$2240 + \$268.80 = \$2508.80.

That is, there is \$2508.80 after two years.

Problems such as the one above are best handled using the formula for the general term of a geometric sequence. This is because the principal every year is being multiplied by the same fixed number.

If the interest rate is r%, the common ratio is $\left(1+\frac{r}{100}\right)$. Explanation of this result:

New principal = old principal + r% of old principal

$$= 1 \times \text{old principal} + \frac{r}{100} \times \text{old principal}$$

$$= \left(1+\frac{r}{100}\right) \times \text{old principal}$$

Instead of using the general term formula $t_n = a.\, r^{n-1}$, we use:

$$A = P\left(1+\frac{r}{100}\right)^n$$

where P = the original amount invested (principal)
r = the interest rate
n = the number of years for the investment
A = the new principal, including interest, at the end of the term.

Example

\$4500 is invested at 6% compound interest calculated yearly. Calculate the amount invested after four years.

Answer

Substitute into $A = P\left(1+\frac{r}{100}\right)^n$ to get:

$$A = 4500 \times \left(1+\frac{6}{100}\right)^4$$
$$= 4500 \times (1.06)^4$$
$$= \$5681.15$$

Inflation

Inflation occurs when prices of items rise. If prices rise at a steady rate, we have what is called the **rate of inflation.**

The approach is the same as for compound interest. Instead of the original amount invested (the principal), we have the original price of an article; and instead of the interest rate, we have the inflation rate. The inflation rate is usually calculated 'per annum', meaning yearly.

The formula for calculating increased prices is:

$$\text{increased price} = \text{old price} \times \left(1+\frac{r}{100}\right)^n$$

where r = inflation rate, and
n = number of years.

Note that this formula is really the same as the compound interest one.

Example

Find the price of a section of land originally worth \$50 000 at the end of a period of 10 years, assuming a constant rate of inflation of 7%.

Answer

$$\text{Price} = \text{old price} \times \left(1+\frac{r}{100}\right)^n$$
$$= 50\,000 \times \left(1+\frac{7}{100}\right)^{10}$$
$$= 50\,000 \times (1.07)^{10}$$
$$= \$98\,000 \quad \text{(to the nearest thousand dollars)}$$

Depreciation

Depreciation occurs when the value of an item decreases. If the value falls at a steady rate, we have what is called the **rate of depreciation**.

The approach is the same as for inflation or compound interest. Instead of the original price or amount invested, we have the original value of an item; and instead of the interest rate or inflation rate, we have the depreciation rate. Like the inflation rate, the depreciation rate is usually calculated 'per annum', meaning yearly.

Because values *decrease* under depreciation, we *subtract* the percentage decrease from the original value.

> The formula for calculating decreased values is
>
> $$\text{decreased value} = \text{original value} \times \left(1 - \frac{r}{100}\right)^n$$
>
> where r = depreciation rate, and
>
> n = number of years.

Example

Find the value (for tax purposes) of a computer originally purchased for $3000 in 2000 at the end of a period of 15 years, assuming the IRD's rate of 20% depreciation for office equipment.

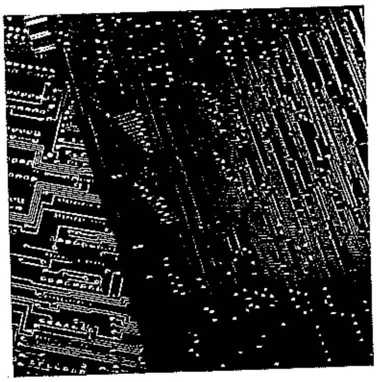
Answer

$$\text{decreased value} = \text{original value} \times \left(1 - \frac{r}{100}\right)^n$$

$$= 3000 \times \left(1 - \frac{20}{100}\right)^{15}$$

$$= 3000 \times (0.80)^{15}$$

$$= \$105.55$$



EXERCISE 15.1

The rates in this exercise are 'per annum'.

1 Calculate the value of these investments placed on the given terms at compound interest:
(a) $5000 at 5% for 8 years
(b) $11 000 at 7.2% for 3 years
(c) $15 000 at 11 % for 7 years
(d) $12 000 at $8\frac{3}{4}$% for 5 years
(e) $20 000 at 3.5% for 18 months

2 Find the *increase in value* of these amounts when placed on compound interest on the given terms:
(a) $16 000 at 12.5% for 5 years
(b) $4000 at 16% for 4 years
(c) $2000 at $22\frac{1}{2}$% for 10 years
(d) $7000 at 18.3% for 15 years
(e) $25 000 at 8% for 30 months

3 Calculate the decreased value of these items given the rates of depreciation below:
(a) An item worth $4600 depreciates at 10% over a period of 6 years.
(b) An item worth $21 000 depreciates at 7% over a period of 4 years.
(c) An item worth $150 000 depreciates at 30% over a period of 5 years.
(d) An item worth $85 000 depreciates at $15\frac{1}{2}$% over a period of 20 months.

EXERCISE 15.2 Applications

1 The population of a town increases by $2\frac{1}{2}$% per annum. If the population in 1994 was 16 000, what was the population in 2002?

2 A car bought for $26 000 depreciates by 5% per year. What is its value after 3 years?

3 The profits of an engineering firm increase by 6% per annum. If the profit in 1983 was $1500, calculate the profit in 2002.

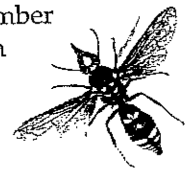
4 The population of a particular Pacific Island nation is predicted to decrease at a constant rate of 3% per year, due to emigration and the death rate exceeding the birth rate. If the population was 65 000 in 2000, estimate the population to the nearest thousand 20 years later.

5 The height of a conifer increases at 12% per year. Calculate the height of a conifer at 4 years of age if it was one metre high when planted.

6 A school roll started at 700 and grew at a rate of 3% for the next 10 years. What was the total roll at the end of this time?

7 A student realised that in Year 7 he needed to do one hour's homework a night to keep up with his studies. Each year he found that he needed to increase his homework time by 20%. What time would he be spending on homework each night in Year 13?

8 A microbiologist believes that it is possible to reduce the number of wasps infesting an area of native bush by releasing a large number of sterile wasps. She estimates that this programme will reduce the number of wasps by 60% per year on an on-going basis. If there are initially 14 000 000 wasps in the area, how many will remain at the end of a 4-year period, assuming that the scientist's calculations are correct?



9 Rip-off Real Estate develops a subdivision of sections for sale to the general public. When the sections are initially sold, the buyers are unaware of an illegal waste disposal site nearby. This subsequently causes the value of sections in the subdivision to decrease at a steady 15% per annum as public concern increases. Find the loss in value (after a 6-year period) on a section initially sold for $85 000.

10 Calculate the total interest earned when:
   (a) $2000 is placed on deposit for 4 years at 12% per annum compounded annually
   (b) $2000 is placed on deposit for 4 years at 3% per quarter (i.e. three months) compounded quarterly.
   (c) Which investment, (a) or (b), earns more?

SPREADSHEET INVESTIGATION – Business depreciation

There are two commonly-used methods of calculating depreciation in business:

- **straight-line** depreciation which involves subtracting a fixed amount from the original value each year. For example:
  straight-line depreciation of 10% on $6000 gives: < $6000, $5400, $4800, ... >
- **diminishing value** depreciation which involves multiplying the original value by a given percentage each year, and subtracting this result from the original value. This gives a new value, which is in turn multiplied by the same percentage. For example:
  diminishing value depreciation of 10% on $6000 gives: < $6000, $5400, $4860, ... >

Two computers are to be depreciated for tax purposes over an indefinite period. Computer A is originally worth $30 000, and is depreciated at 10% using the straight-line method. Computer B is originally worth $25 000, and is depreciated at 10% using the diminishing value method.

1 Which method gives an arithmetic sequence?
2 Which method gives a geometric sequence?
3 Produce a spreadsheet showing what happens to the value of the computers over a 10-year period.
4 After how many years is computer B valued higher than computer A?

Rearrangements of the growth/decay formula

Variations on growth and decay problems involve rearrangements of the formula.

Note that in the compound interest formula $A = P\left(1 + \frac{r}{100}\right)^n$ there are four variables involved:

A, P, r and n.

In any compound interest problem, *three* of these variables must be provided in order to find the fourth.

Example

Use the compound interest formula $A = P\left(1 + \frac{r}{100}\right)^n$ to calculate:

(a) P, given $A = \$4259.20$, $r = 10\%$ and $n = 3$

(b) r, given $A = \$20\,000$, $P = \$5000$ and $n = 20$

(c) n, given $A = \$7276.25$, $P = \$3500$ and $r = 5\%$.

Answer

(a)
$$4259.2 = P\left(1 + \frac{10}{100}\right)^3$$
$$4259.2 = P \times (1.1)^3$$
$$4259.2 = 1.331P$$
$$P = \frac{4259.2}{1.331} = \$3200$$

(b)
$$20\,000 = 5000\left(1 + \frac{r}{100}\right)^{20}$$
$$\frac{20\,000}{5000} = (1 + 0.01r)^{20}$$
$$4 = (1 + 0.01r)^{20}$$

Now take the 20th root of each side (use a calculator):

$$\sqrt[20]{4} = 1 + 0.01r$$
$$1.071\ 77 = 1 + 0.01r$$
$$0.071\ 77 = 0.01r$$
$$r = \frac{0.071\ 77}{0.01} = 7.177\%$$

(c)
$$7276.25 = 3500\left(1 + \frac{5}{100}\right)^n$$
$$7276.25 = 3500 \times (1.05)^n$$
$$(1.05)^n = \frac{7276.25}{3500} = 2.078\,928\,6$$

Now take logs of both sides:

$$\log(1.05)^n = \log(2.078\,928\,6)$$
$$n\log(1.05) = \log(2.078\,928\,6)$$
$$n = \frac{\log(2.078\,928\,6)}{\log(1.05)}$$
$$= \frac{0.317\,839\,6}{0.021\,189\,3}$$
$$= 15$$

When working through problems similar to this, it is not necessary to write down all the calculator values. However, you should work to full-figure accuracy, and only round answers at the end of the problem.

EXERCISE 15.3

1 Calculate (to the nearest cent) the original amount invested, given the compound interest terms below.

(a) What principal amounts to \$15 000 at 11% for 6 years?

(b) What principal amounts to \$80 500 at 8.5% for 5 years?

(c) What principal amounts to \$150 000 at 14% for 21 months?

2 Find the rate of compound interest (to 2 dp) needed to produce these results on investments for the given number of years:

(a) \$500 increases to \$1500 after 10 years

(b) \$1400 increases to \$2150 after 3 years

(c) \$80 000 increases to \$105 000 after 6 years.

3 Calculate the number of years needed to produce these results on compound interest investments at the given rates. Give answers to 2 dp.

(a) \$8000 increases to \$15 000 invested at 12% per annum

(b) \$250 increases to \$1300 invested at $7\frac{1}{2}$% per annum

(c) \$25 750 increases to \$72 300 invested at 6.2% per annum

4 Calculate the rate of inflation given the information below:

(a) Goods bought for \$175 in 1978 cost \$625 in 1988

(b) Goods bought for \$36 in 1973 cost \$86 in 1991

5 A car bought for \$32 000 depreciates in value to \$18 500 after 3 years. Calculate the annual rate of depreciation over the period.

6 A house purchased 16 years ago was sold for \$230 000 recently. The owner calculated that the house had gained value at a steady rate of 9% per year. What did the owner pay for the house herself?

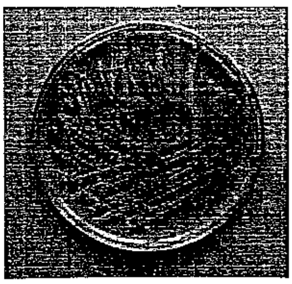
7 An electric motor loses efficiency at a steady rate of 4% per year as its components deteriorate. How many years does it take for it to decline from a rating of 2500 watts to 2160 watts?

8 Aristotle Holdings invested in a fleet of yachts three years ago. Recently they had to sell the yachts at \$1 500 000 each, representing a loss of 4% per annum to the company. What was the original price of each yacht?

9 A scientist grows bacteria in a Petri dish. The number of bacteria grows by 50% each hour. Initially there are 20 bacteria in the dish.

(a) How many bacteria are present after 6 hours?

(b) After how many hours will there be 12 000 bacteria?



10 A farmer in the Mackenzie Country knows that without pest control, the number of rabbits on his property increases at a constant rate of 2% per week. How many weeks would it take for the number of rabbits to increase from 450 to 1060?

11 A property speculator bought a house for \$150 000. When she tried to resell the house three years later, she discovered that the housing market had declined, and the house was only worth \$98 000. What was the yearly rate of decrease in value?

12 After a steady increase of 15% per year, the share price of Theta Publishing Ltd is now \$1.56 per unit. Calculate the price per unit 8 years ago.

13 Air Atlantis bought a fleet of aircraft at \$7 000 000 per craft. After 10 years they were offered for sale at \$2 500 000 each. What was the percentage rate of depreciation per annum?

14 The price of a share in Delta Enterprises has risen from \$2.30 to \$5.67 over a number of years since an investor first purchased shares in the company. The investor calculates that this rise in value represents an increase of 30% per annum. Find the number of years that the investor has held the shares.

15 Over the last 12 years, there has been a constant decrease of 4% per year in the numbers of visitors to a tourist attraction in Rotorua. If there were 45 000 visitors this year, how many were there six years ago?

16 When a 10-acre block was sold for \$100 000 recently, the owners were delighted: after a period of 5 years their investment had returned an interest rate compounded at 14% per year. How much had the land cost originally?

17 Town planners know there has been a constant increase of 5% per year in the number of commuters using public transport at least once a week in a large city. This increase is projected to continue at only 3% per year. If there were 67 000 commuters using public transport six years ago, how many should be using it in seven years time?

...ming a constant rate of inflation over the ...iod, calculate the price of goods in 1998 ...en that they were worth $560 in 1979 and ...40 in 1985.

...business has a laser printer which was ...rchased for $6500. The business prints about 400 000 pages per year, and the printer decreases in value by 30% each year. It is expected to print 1 500 000 pages of paper before it is replaced. What will it be worth then?

20 A granary in Canterbury stores wheat in a large silo. They lose a constant percentage per month due to rodent spoilage and mildew. At the beginning of April one year there were 45 000 tonnes stored, but at the beginning of September there were only 36 700 tonnes stored. How much wheat was lost in the month of July?

21 Each time a paint brush is rinsed in a container of fresh water about 96% of the paint is removed. How many times will the brush need to be rinsed before the amount of paint remaining is less than 0.001% of the amount of paint originally in the brush?

SPREADSHEET INVESTIGATION – Adding to the original investment

Sometimes a person can make regular deposits into a savings account that is earning compound interest. These deposits earn interest as well.

The spreadsheet below shows how to calculate the amount in the account at the end of each year:

- The initial amount is in cell B1
- The regular yearly deposit is in cell E1
- The interest rate is in cell G1.

This extract shows the formulas needed in the first few rows. To continue the calculations over a longer period, copy the formulas in B5 and C5 downwards a number of times.

	A	B	C	D	E	F	G
1	Initial amount =			Regular deposit =		Interest rate =	
2							
3			Interest earned				
4	At beginning	=B1	=B4*G$1/100				
5	End of 1st year	=B4+C4+E$1	=B5*G$1/100				
6	End of 2nd year	=B5+C5+E$1	=B6*G$1/100				

Notice that we use a $ symbol in the formula for E1. That is, E$1 will keep the cell fixed as E1 when copied downwards, rather than giving E2, E3, etc.

This spreadsheet shows the result after four years when someone invests $2000 at 8% interest and adds regular deposits of $150 each year.

	A	B	C	D	E	F	G
1	Initial amount =	$2,000.00		Regular deposit =	$150.00	Interest % =	8
2							
3			Interest earned				
4	At beginning	$2,000.00	$160.00				
5	End of 1st year	$2,310.00	$184.80				
6	End of 2nd year	$2,644.80	$211.58				
7	End of 3rd year	$3,006.38	$240.51				
8	End of 4th year	$3,396.89	$271.75				

EXERCISE 15:4 Applications

Use a spreadsheet to answer these questions.

1 A student invested her earnings from a Saturday job at a rate of 8% per annum. If she invested $800 at the beginning of the first year, and each year thereafter invested a further $800, how much did she have in her account at the end of 5 years?

2 A father provided for his son when he was born by investing $1000 in a savings account at a constant interest rate of 12% per annum. Each year he added a further $500. How much would there be in the account at the end of 12 years when his son was due to start secondary school?

3 A special form of endowment insurance requires an initial investment of $500 and then further deposits of $200 each year. If the interest is compounded annually at 6%, what is the total value of the investment after a period of 10 years?

4 Create a spreadsheet which calculates the total value when:

- an initial amount is invested in an account that pays interest compounded annually
- the interest rate is fixed over the whole period
- different amounts are added as deposits to the account at the end of each year.

Use your spreadsheet to calculate the amount in this savings account on January 1, 2006. The savings account pays 5% interest, and deposits are made as follows:

Date	Deposit
Jan 1, 2000 (initial deposit)	$4000
Jan 1, 2001	$2700
Jan 1, 2002	$3100
Jan 1, 2003	$650
Jan 1, 2004	$5111
Jan 1, 2005	$810

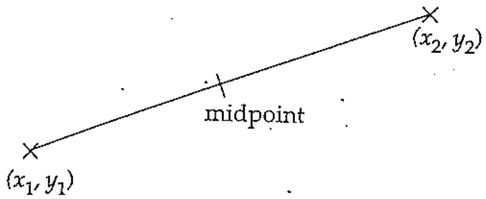
15

16 Co-ordinate geometry

Co-ordinate geometry is one of the most interesting branches of Mathematics, because it links together algebra and geometry. It combines the skill of manipulating formulae and equations, together with understanding two-dimensional relationships.

Midpoint of a line segment

A **line segment** is the part of a line that lies *between* two points. The **midpoint** of the line segment is the point that lies exactly halfway between the points:



In general, if the two points are (x_1, y_1) and (x_2, y_2) then the co-ordinates of the midpoint are:

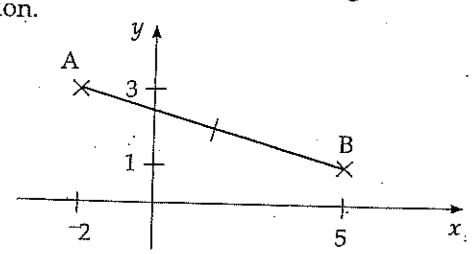
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Example

Find the midpoint of the line segment AB, where A = $(^-2, 3)$ and B = $(5, 1)$.

Answer

First draw a diagram to get a feeling for this question.



$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$= \left(\frac{^-2 + 5}{2}, \frac{3 + 1}{2}\right)$$

$$= (1.5, 2)$$

EXERCISE 16.1

1 Write down the co-ordinates of the midpoints of the line segments joining:

(a) $(4, 2)$ and $(8, 6)$
(b) $(3, 4)$ and $(7, 2)$
(c) $(^-1, 3)$ and $(^-5, 1)$
(d) $(^-8, 2)$ and $(^-1, 4)$
(e) $(^-2, ^-3)$ and $(4, 3)$
(f) $(1, 2)$ and $(1, 5)$
(g) $(6, ^-1)$ and $(^-2, ^-3)$
(h) $(^-7, 4)$ and $(2, 4)$

2 What are the co-ordinates of the midpoints of the line segments joining:

(a) $(2.9, ^-4.7)$ and $(^-0.8, 5.3)$
(b) $\left(\frac{3}{4}, \frac{5}{6}\right)$ and $\left(\frac{^-1}{2}, \frac{1}{3}\right)$

3 Express the co-ordinates of the midpoints of the line segments joining these points as simply as possible:

(a) $(p + q, 2p)$ and $(p - q, 0)$
(b) $(a + 1, a - 2)$ and $(a - 1, a + 2)$
(c) $(4f + 5, 2g - 1)$ and $(6f - 1, ^-12g + 3)$

4 The co-ordinates of the midpoint of the line segment AB are $(^-2, 4)$. If A = $(6, ^-1)$, what are the co-ordinates of B?

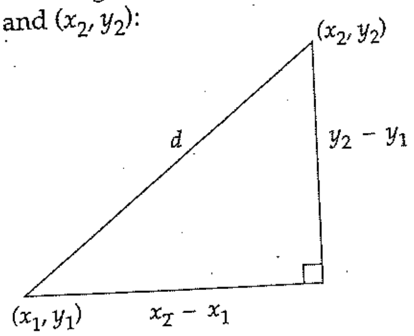
5 AB is the diameter of a circle. If A = $(2, ^-6)$ and B = $(4, 2)$, write down the co-ordinates of the centre of the circle.

6 A circle has centre $(4, ^-5)$. If a diameter is PR and P = $(8, 1)$, calculate the co-ordinates of R.

Distance between two points

Given two points on a graph it is always possible to find the distance between them using Pythagoras.

In general the given points are (x_1, y_1) and (x_2, y_2):



These are joined up. The distance (d) between them can be calculated by finding the length of the hypotenuse of a right-angled triangle:

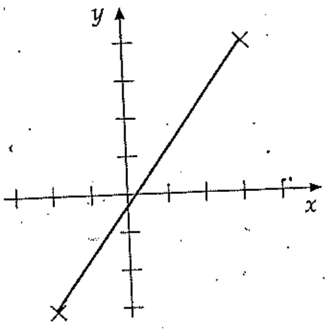
$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example

Calculate the distance between the points $(^-2, ^-3)$ and $(3, 4)$.

Answer



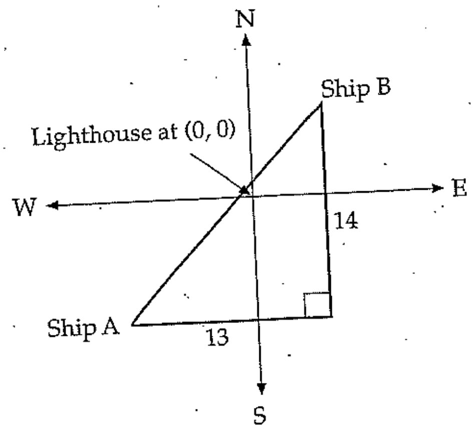
$$d^2 = (3 - {^-2})^2 + (4 - {^-3})^2$$
$$= 5^2 + 7^2$$
$$= 25 + 49$$
$$= 74$$
$$d = \sqrt{74} = 8.602 \text{ (4 sf)}$$

Example

Ship A is 10 km south and 8 km west of a lighthouse. Ship B is 4 km north and 5 km east of the same lighthouse. What is the shortest distance between the two ships?

Answer

Take the lighthouse as being at the origin (0, 0). Apply the directions N, E, S, W to the axes. Then ship A is at $(^-8, ^-10)$ and ship B is at $(5, 4)$.



Distance d is given by:

$$d^2 = (5 - {^-8})^2 + (4 - {^-10})^2$$
$$= 13^2 + 14^2$$
$$= 169 + 196$$
$$= 365$$
$$d = \sqrt{365} = 19.105$$
$$= 19 \text{ km (to nearest km)}$$

EXERCISE 16.2

1 Calculate the distance between these pairs of points (to 4 sf if answer not exact):

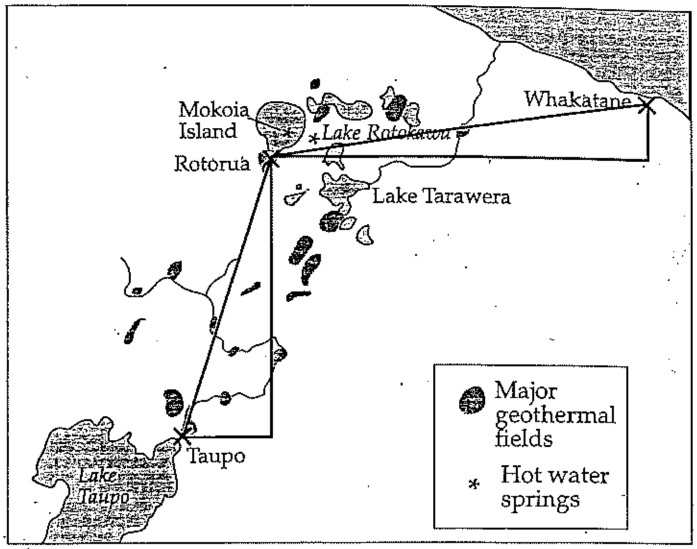
(a) (3, 1) and (6, 5)
(b) (0, 0) and (1, 1)
(c) $(^-2, ^-3)$ and $(3, ^-15)$
(d) $(6, ^-2)$ and $(4, ^-3)$
(e) (2, 8) and $(^-1, ^-5)$
(f) (12, 7) and (12, 9)
(g) $(^-3, 5)$ and (2, 5)
(h) $(^-3, ^-3)$ and $(^-2, ^-2)$

2 Use the distance formula to calculate the distance between these pairs of points:

(a) (6.1, 8.3) and (0.7, 1.9) (b) ($^-$4.6, $^-$0.7) and (9.1, $^-$6.5) (c) $\left(\frac{5}{3}, \frac{3}{4}\right)$ and $\left(\frac{2}{3}, \frac{1}{2}\right)$

3 What is the distance of ($^-$7, 24) from the origin?

4 Taupo is 20 km west and 62 km south of Rotorua. Whakatane is 82 km east and 12 km north of Rotorua. Calculate the distance between Taupo and Whakatane to the nearest km.



5 A traveller is lost in a desert. Unknown to him, he is 4 km west and 5 km south of an oasis.

A helicopter is searching for him and is currently at a point 3 km east and 1 km south of the oasis.

Calculate the distance between the traveller and the helicopter.

6 Show that $\triangle PQR$ with vertices P(2, 7), Q($^-$1, 1) and R(5, 1) is isosceles. Name the pair of equal sides.

7 ABCD is a parallelogram, where A = (1, 1), B = (5, 2), C = (6, 3) and D = (2, 2). Calculate the length of the longest diagonal of ABCD.

8 The four points ABCD form a square where A = (1, 1), B = (6, 3), C = (4, 8) and D = ($^-$1, 6). A circle passes through all four points.

(a) Calculate the perimeter of the square ABCD.

(b) Calculate the radius of the circle.

Gradients

Gradient of the line joining two given points

The **gradient** of a line is a number which describes how steep the line is. It is calculated from the fraction:

$$\frac{\text{change in } y}{\text{change in } x}$$

In general, the formula or rule giving the gradient of the line joining two points (x_1, y_1) and (x_2, y_2) is:

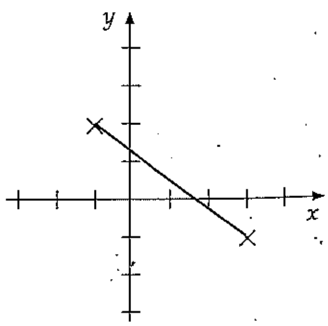
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Example

Find the gradient of the line joining ($^-$1, 2) and (3, $^-$1).

Answer

First draw a diagram to get a feeling for this question.



$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{^-1 - 2}{3 - ^-1} = \frac{^-3}{4}$$

We say that the gradient of a *vertical* line is **undefined**. The *x*-coordinates of any two points on a vertical line are the same. The gradient formula would involve dividing by 0, which is undefined.

$$\text{Gradient} = \frac{r - q}{p - p} = \frac{r - q}{0}$$

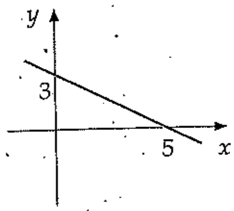
EXERCISE 16.3

1 Calculate the gradient of the line passing through the pair of given points. Leave your answer as a fraction in its simplest form.

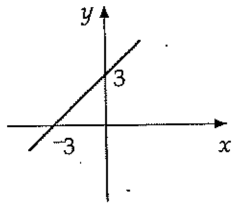
(a) (4, 2) and (3, 5)
(b) (2, 7) and (1, 4)
(c) ($^-1$, 5) and (4, 7)
(d) (2, $^-1$) and ($^-3$, 6)
(e) ($^-4$, 3) and ($^-5$, $^-2$)
(f) ($^-9$, $^-3$) and ($^-1$, $^-4$)

2 Calculate the gradients of the lines drawn in the diagrams below:

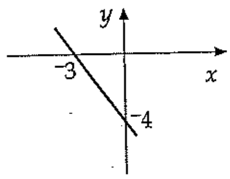
(a)



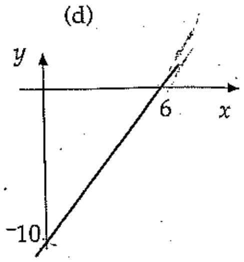
(b)



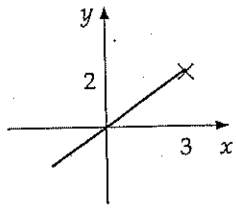
(c)



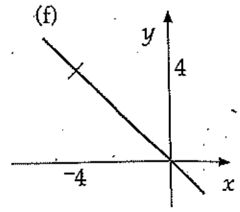
(d)



(e)



(f)



3 Lines pass through the following pairs of points. Calculate their gradients.

(a) (0, 0) and (9, 1)
(b) (23, 78) and (45, $^-65$)
(c) (27, 40) and (27, 56)
(d) ($^-8$, 7) and ($^-1$, 7)
(e) (4.1, 6.7) and (9.5, $^-2.3$)

4 The line joining (*a*, 7) to (3, 4) has a gradient of 1. Form an equation and solve it to work out the value of *a*.

5 The gradient of the line joining (3, *p*) to ($^-1$, 2) is $\frac{^-1}{4}$. Calculate the value of *p*.

6 Calculate the gradients of the lines joining these pairs of points. Simplify your answers if possible.

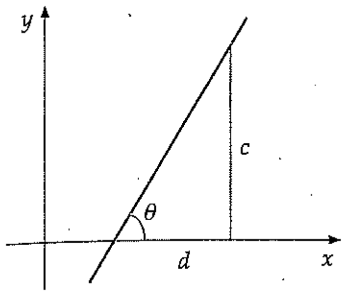
(a) $(c - 2, c + 3)$ and $(c + 1, c - 4)$
(b) $(2d - 4, 3d + 6)$ and $(3d - 1, d + 5)$
(c) (p, q) and (q, p)
(d) (p, q) and $(^-p, ^-q)$
(e) (p, q) and (r, q)
(f) (p, q) and (p, r)

7 A triangle ABC has vertices A (4, $^-2$), B ($^-1$, 3) and C (4, $^-6$). Calculate the gradient of the line joining the midpoints of AB and AC.

8 The gradient of the line joining (1, 3) to (*p*, 4) is $p - 1$. Solve a quadratic equation to calculate the two possible values for *p*.

Gradient of the line making an angle of θ with the x-axis

The following diagram shows a line that makes an angle of θ with the positive direction of the x-axis.



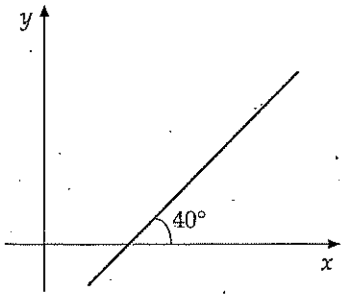
The gradient of this line is the fraction $\frac{c}{d}$.

But from trigonometry $\frac{c}{d} = \tan\theta$

It follows that if we know the angle that a line makes with the positive direction of the x-axis, the gradient of the line will be tan of this angle. And if we know the gradient of a line, we can use trigonometry ($\tan^{-1}$) to calculate the angle between the line and the x-axis.

Example

A line makes an angle of 40° with the x-axis, as shown in the diagram. Calculate the gradient of the line.

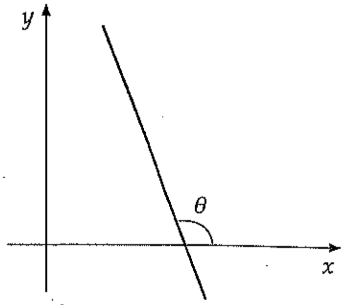


Answer

Gradient = tan 40° ≐ 0.84

Example

The line in the diagram has gradient $^-3$. Calculate the angle between the line and the positive direction of the x-axis. (Note the gradient is negative. This means the angle is obtuse, i.e. greater than 90°.)



Answer

$m = {}^-3$

$\tan\theta = {}^-3$

$\theta = {}^-71.6°$ (using calculator)

The required angle is 108.4°, obtained by adding 180° to the calculator value. In fact, the calculator gives the angle *below* the x-axis.

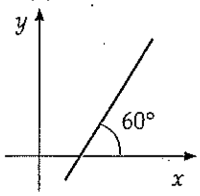
EXERCISE 16.4

1 Calculate the gradient of a line making the following angles with the positive direction of the x-axis:

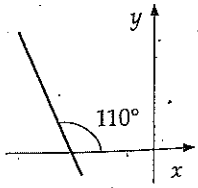
(a) 24° (d) 135°
(b) 30° (e) 180°
(c) 126° (f) 90°

2 Calculate, to 2 sf, the gradient for each of these lines.

(a)

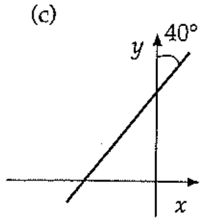


(b)



continues...

(c)



3 Calculate the angles that the lines with these gradients make with the positive direction of the x-axis.

(a) 2

(b) $\frac{1}{4}$

(c) $\frac{^{-}2}{3}$

(d) 1

(e) 0

(f) $\sqrt{3}$

(g) $^{-}1$

(h) $\frac{^{-}1}{\sqrt{3}}$

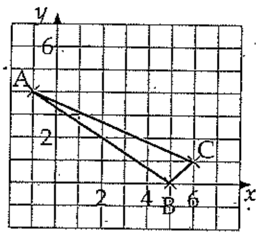
(i) undefined

4 What angle does the line joining P(4, 3) and Q($^{-}2$, 0) make with the positive direction of the x-axis?

5 Calculate the acute angle formed when the line joining (8, $^{-}1$) and ($^{-}3$, 4) cuts the x-axis.

6 Calculate the acute angle formed when the line joining ($^{-}2$, 3) and (1, 4) cuts the y-axis.

7 The triangle ABC has vertices A = ($^{-}1$, 4), B = (5, 0) and C = (6, 1).



By first finding the angles between AB and the x-axis, and BC and the x-axis, calculate the size of $\angle$ABC.

Equations of lines

Any relation in the form

$$y = mx + c$$

gives a straight line when graphed. Such a relation is called **linear**. The two numbers m and c give special information about the line:

- m gives the **gradient** of the line
- c gives the **y-intercept**, the point where the line cuts the y-axis.

Rearranging line equations to the form $y = mx + c$

The above information is only obvious if the straight-line equation is already in this form; i.e. if y is already the 'subject' of the equation. The following example shows how to rearrange a straight-line equation to obtain information about the gradient.

Example

Calculate the gradient of the line $5x - 3y + 2 = 0$. Hence find the angle the line makes with the positive direction of the x-axis.

Answer

Make y the subject—i.e. aim to get y by itself on the left-hand side of the equals sign.

$$5x - 3y + 2 = 0$$

$$^{-}3y = {^{-}5x} - 2$$

$$y = \frac{^{-}5x}{^{-}3} - \frac{2}{^{-}3}$$

$$y = \frac{5x}{3} + \frac{2}{3}$$

This means the gradient is $\frac{5}{3}$.

The angle between this line and the positive direction of the x-axis is found by solving $\tan\theta = \frac{5}{3}$.

$$\tan\theta = \frac{5}{3} = 1.\dot{6}$$

$$\theta = 59.0° \quad (\tan^{-1} \text{ on a calculator})$$

EXERCISE 16.5

1 Find the gradients of these lines by rearranging them to the form $y = mx + c$ first:

(a) $2y = 3x - 2$

(b) $3y = x - 6$

(c) $x + y = 3$

(d) $2x + y = 3$

(e) $4x + 5 = 2y$

(f) $3x - 2y = 6$

(g) $4x - y = 8$
(h) $x - 4y = 2$
(i) $4x + 2y + 9 = 0$
(j) $4y = 2x - 8$
(k) $x + 3y - 2 = 0$
(l) $4x = 2 + 3y$

2 Find the gradient of the line $ax + by + c = 0$.

3 Calculate the angle that each of these lines makes with the positive direction of the x-axis:

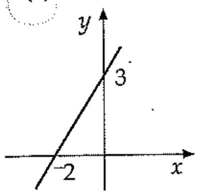
(a) $y = 2x - 1$
(b) $y = \frac{x}{3} + 5$
(c) $y = {}^-4x + 12$
(d) $y = \frac{{}^-3x}{5} - 7$
(e) $y = x$
(f) $y = {}^-x$

4 Calculate the angle that each of these lines makes with the positive direction of the x-axis:

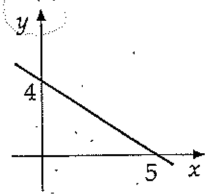
(a) $3x - 4y = 5$
(b) $2x + y = 8$
(c) $x - 3y + 5 = 0$
(d) $4x + 5y - 2 = 0$

5 Calculate the angles that each of the lines in the diagrams below makes with the positive direction of the x-axis.

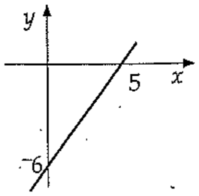
(a)



(b)



(c)



The form $ax + by + c = 0$

Although the form $y = mx + c$ is particularly convenient when working with gradients, not all straight lines can be expressed in this form; vertical lines cannot be. However, all straight lines *can* be written in the form $ax + by + c = 0$.

At this level of Mathematics there is an increasing emphasis on style and elegance. The convention we follow is that it is best to express straight line equations in this form, with a, b and c as integers, so that the equation has no fractions. The x term should be written first, and with a positive coefficient (number multiplying x).

Example

Express $y = 2x - 5$ in the form $ax + by + c = 0$.

Answer

$$y = 2x - 5$$
$${}^-2x + y + 5 = 0$$
$$2x - y - 5 = 0$$

Example

Express $y = \frac{{}^-2x}{3} + 4$ in the form $ax + by + c = 0$.

Answer

$$y = \frac{{}^-2x}{3} + 4$$
$$3y = {}^-2x + 12$$
$$2x + 3y - 12 = 0$$

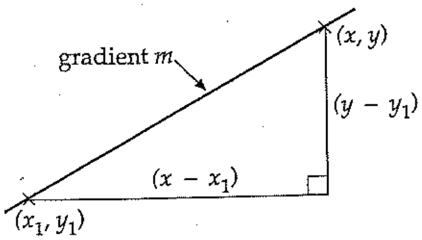
EXERCISE 16.6

Express these straight-line equations in the form $ax + by + c = 0$.

1 $y = x + 4$
2 $y = 2x - 7$
3 $y = {}^-3x + 1$
4 $y = x$
5 $x = 4$
6 $y = \frac{2x}{7} - 5$
7 $y = \frac{{}^-3x}{4} + 1$
8 $x + y = 4$
9 $y = 4x + \frac{2}{5}$
10 $y = \frac{{}^-x}{4} - \frac{3}{5}$
11 $\frac{2x}{3} = \frac{y}{4}$
12 $\frac{2y}{5} = \frac{4x}{3} - \frac{1}{2}$

Vriting the equations of lines rom given information

Consider a line that passes through the fixed oint (x_1, y_1). The line has gradient m. Any other eneral point on the line is (x, y).



There are two possible expressions for the radient:

- the fraction $\dfrac{\text{change in } y}{\text{change in } x}$, OR
- the given gradient m.

These two expressions must be equal. Therefore:

$$\boxed{\begin{array}{ll} & \dfrac{y - y_1}{x - x_1} = m \\ \text{OR} & y - y_1 = m(x - x_1) \end{array}}$$

There are several different ways in which you can be given information about straight lines. Three of these ways are given below. In each case, you have to determine the equation of the line.

1 Given gradient m and a point (x_1, y_1) on the line

The equation of the line is:

$$y - y_1 = m(x - x_1)$$

This is often called the **point/gradient equation.**

Example

Find the equation of the line that passes through $(4, ^-3)$ with gradient 2.

Answer

$$y - y_1 = m(x - x_1)$$
$$y - ^-3 = 2(x - 4)$$
$$y + 3 = 2x - 8$$
$$y = 2x - 11$$
$$\text{OR} \quad 2x - y - 11 = 0$$

EXERCISE 16.7

1–7 *Find the equation for each of the following lines. Express your answer in the form $ax + by + c = 0$.*

1 Through (2, 5) with gradient 3.

2 Through $(^-1, 4)$ with gradient 2.

3 Through $(3, ^-6)$ with gradient $\dfrac{1}{2}$.

4 Through (5, 1) with gradient $^-4$.

5 Through $(^-2, ^-3)$ with gradient $\dfrac{1}{4}$.

6 Through $(^-3, 4)$ with gradient $\dfrac{-2}{5}$.

7 Through $(^-6, 1)$ with gradient 0.

8 Find the equation of the line with gradient $^-2$ which passes through the midpoint of PQ where P = (4, 6) and Q = $(^-2, 4)$.

2 Given two points (x_1, y_1) and (x_2, y_2) on the line

In this case the gradient m is: $\dfrac{y_2 - y_1}{x_2 - x_1}$

Calculate this first, then use the formula in **1** above.

Example

Find the equation of the line joining the points $(^-2, 1)$ and (3, 2).

Answer

$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 1}{3 - ^-2} = \frac{1}{5}$$

Now substitute into $y - y_1 = m(x - x_1)$:

$$y - 1 = \frac{1}{5}(x - ^-2)$$
$$5y - 5 = x + 2 \quad \text{(cross-multiplying and simplifying)}$$
$$5y - x - 7 = 0$$
$$x - 5y + 7 = 0$$

This is often called the **point/point** equation.

An alternative expression for this equation is $\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$

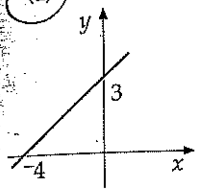
XERCISE 16.8

Find the equation of the line joining each of these pairs of given points:

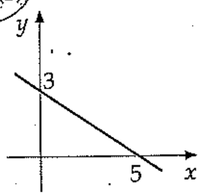
(a) $(4, 1)$ and $(7, 3)$
(b) $(5, 4)$ and $(7, 1)$
(c) $(3, ^{-}2)$ and $(2, 7)$
(d) $(4, 3)$ and $(^{-}1, ^{-}2)$
(e) $(6, ^{-}2)$ and $(1, ^{-}4)$
(f) $(^{-}4, 3)$ and $(2, ^{-}4)$
(g) $(^{-}5, 1)$ and $(^{-}3, ^{-}2)$
(h) $(4, 1)$ and $(4, ^{-}6)$
(i) $(3, ^{-}2)$ and $(1, ^{-}2)$

2 Write down the equation of the lines drawn in the diagrams below:

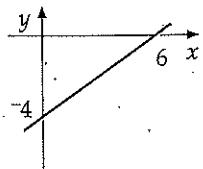
(a)



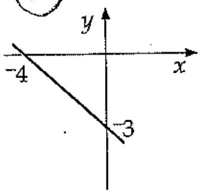
(b)



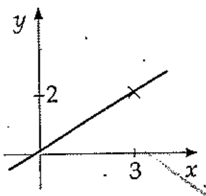
(c)



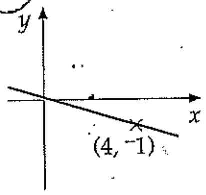
(d)



(e)



(f)



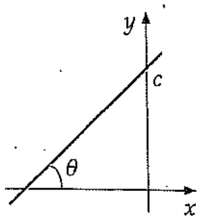
3 The points P $(^{-}1, 3)$ and Q $(4, 6)$ lie on the circumference of a circle. What is the equation of the chord joining them?

4 A line drawn through the point $(^{-}4, 5)$ touches a circle at $(2, ^{-}1)$. Find the equation of this tangent.

5 Find the equation of the line that cuts the x-axis at 4 and the y-axis at $^{-}2$.

6 The points $(2, 1)$, $(^{-}3, q)$ and $(1, ^{-}2)$ are **collinear**—that is, they lie on the same line. Calculate the value of q.

3 Given the angle θ with the x-axis and a point on the line



If the angle that the line makes with the positive direction of the x-axis is θ, then the **gradient** of the line must be **tan θ**. Use this information to substitute into the equation $y - y_1 = m(x - x_1)$.

Example

Find the equation of the line that has a y-intercept of 5 and crosses the x-axis at an angle of 45°.

Answer

The gradient of the line = tan 45° = 1. The line passes through $(0, 5)$. Substitute into $y - y_1 = m(x - x_1)$.

$$y - 5 = 1(x - 0)$$
$$y - 5 = x$$
$$y = x + 5$$

OR $\quad x - y + 5 = 0$

EXERCISE 16.9

1 In each of the following, find the equation of the line which passes through the given point and intersects with the positive direction of the x-axis at the given angle.

(a) $(3, 4)$ 45°
(b) $(6, ^{-}1)$ 0°
(c) $(^{-}2, 3)$ 135°
(d) $(^{-}5, ^{-}6)$ 45°
(e) $(5, ^{-}8)$ 135°

2 Write down the equation of each line that passes through the given point and intersects with the positive direction of the x-axis at the given angle. Where necessary, round all numbers in the final equation to 2 dp.

(a) $(4, 1)$ 60°
(b) $(^-2, ^-1)$ 30°
(c) $(8, ^-3)$ 167°
(d) $(0, ^-2)$ 10°
(e) $(6, 0)$ 180°

3 The hypotenuse of a right-angled isosceles triangle lies on the x-axis, and the vertex opposite the hypotenuse is the point $(4, 5)$. Find the equations of the two equal sides.

Parallel lines

Parallel lines have the same direction. This means they never meet.

Parallel lines (in two-dimensions) have the *same gradient*.

> Consider the two lines $y = m_1x + c_1$ and $y = m_2x + c_2$.
>
> If $m_1 = m_2$ then the two lines are parallel.

Example

Are the two lines (1) $y = \frac{3x}{4} - 5$ and (2) $4y - 3x + 10 = 0$ parallel?

Answer

By inspection the gradient of line (1) is $\frac{3}{4}$.

Rearranging the equation for line (2) to the form $y = mx + c$, we obtain $4y = 3x - 10$ which also has gradient $\frac{3}{4}$.

So the two lines are parallel.

Example

What is the equation of the line which passes through the point $(6, ^-2)$ and is parallel to the line $2x - 3y + 4 = 0$?

Answer

Rearrange the equation of the given line:

$$2x - 3y + 4 = 0$$
$$^-3y = {}^-2x - 4$$
$$y = \frac{^-2x}{^-3} - \frac{4}{^-3}$$
$$y = \frac{2x}{3} + \frac{4}{3}$$

The gradient of this line is $\frac{2}{3}$. So the required line has gradient $\frac{2}{3}$ and passes through $(6, ^-2)$.

The equation of the line is found by substituting into $y - y_1 = m(x - x_1)$.

$$y - {}^-2 = \frac{2}{3}(x - 6)$$
$$3(y + 2) = 2(x - 6)$$
$$3y + 6 = 2x - 12$$
$$^-2x + 3y + 18 = 0$$
$$2x - 3y - 18 = 0$$

EXERCISE 16.10

1 Here are the equations of 10 lines. Find the gradient for each line and then write down all the sets of parallel lines: for example, if a and c have the same gradient then one set of parallel lines is $\{a, c\}$.

(a) $y = 3x - 5$
(b) $y = 4 - 3x$
(c) $y - 3x = 2$
(d) $3y = x + 6$
(e) $x + 3y = 3$
(f) $3x + 2y = 6$
(g) $6y = 2x + 1$
(h) $2x - 3y = 2$
(i) $y + 3x = 6$
(j) $3x - 2y = 4$

2 Find the equation of the line through $(^-1, 6)$ parallel to the line $y = 3x - 2$.

3 Find the equation of the line through $(3, ^-1)$ parallel to the line $y = {}^-x + 2$.

4 Find the equation of the line through $(7, 2)$ parallel to the line $4x - y + 2 = 0$.

5 Find the equation of the line through $(0, {}^{-}2)$ parallel to:
(a) $4x - 2y + 1 = 0$ (b) $x + 5y - 10 = 0$

6 A triangle is given by A $(3, 2)$, B $({}^{-}1, 5)$ and C $(0, {}^{-}1)$. What is the equation of the line through B parallel to AC?

7 The line through $(1, 2)$ and $(3, a)$ is parallel to the line $2x + y - 5 = 0$. Calculate the value of a.

Perpendicular lines

Two lines are **perpendicular** if they meet at right-angles. The gradients multiply to ${}^{-}1$. Each gradient is the negative reciprocal of the other.

> If the two lines are perpendicular:
>
> $m_1.m_2 = {}^{-}1$
>
> In particular this means $m_1 = \dfrac{{}^{-}1}{m_2}$ and $m_2 = \dfrac{{}^{-}1}{m_1}$

There is one exception to this rule. A horizontal line (gradient 0) and a vertical line (gradient undefined) are perpendicular to each other. In this case $m_1 \times m_2$ has no value.

Example

Find the gradient of a line perpendicular to the line $x + 3y - 4 = 0$.

Answer

The given line can be rearranged to $3y = {}^{-}x + 4$ or $y = \dfrac{{}^{-}1}{3}x + \dfrac{4}{3}$ which has gradient $\dfrac{{}^{-}1}{3}$.
A line perpendicular to this one would have gradient 3.

Note: $\dfrac{{}^{-}1}{3} \times 3 = {}^{-}1$ as required.

Example

Find the equation of the line which passes through $(4, 6)$ and is perpendicular to the line passing through the points $(2, {}^{-}3)$ and $({}^{-}1, 5)$.

Answer

The gradient of the line through $(2, {}^{-}3)$ and $({}^{-}1, 5)$ is given by the formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

That is, $m = \dfrac{5 - {}^{-}3}{{}^{-}1 - 2} = \dfrac{8}{{}^{-}3} = \dfrac{{}^{-}8}{3}$

The gradient of a perpendicular line would be $\dfrac{3}{8}$.

$\left(\text{Note: } \dfrac{3}{8} \text{ is the negative reciprocal of } \dfrac{{}^{-}8}{3}\right)$.

The required line has gradient $\dfrac{3}{8}$ and passes through $(4, 6)$. The equation is:

$$y - 6 = \frac{3}{8}(x - 4)$$
$$8y - 48 = 3x - 12$$
$$3x - 8y + 36 = 0$$

Example

Give the equation of the line perpendicular to $y = 2x + 6$ which has the same x-intercept.

Answer

Two pieces of information are required:
- the gradient, and
- the point that the line passes through.

Gradient $= \dfrac{{}^{-}1}{2}$ (product of gradients is ${}^{-}1$).
The x-intercept for the line is when $y = 0$:

$0 = 2x + 6 \quad x = {}^{-}3$

i.e. x-intercept is $({}^{-}3, 0)$

Substitute into $y - y_1 = m(x - x_1)$ to obtain the equation:

$$y - 0 = \frac{{}^{-}1}{2}(x - {}^{-}3)$$
$$2y = {}^{-}1(x + 3)$$
$$2y = {}^{-}x - 3$$
$$x + 2y + 3 = 0$$

EXERCISE 16.11

odd 1 Give the gradients of lines perpendicular to lines with these gradients:

(a) 2
(b) 7
(c) $\frac{1}{3}$
(d) $\frac{^{-}1}{4}$
(e) $^{-}1$
(f) $\frac{^{-}2}{5}$
(g) $^{-}8$
(h) $\frac{1}{6}$

2 (a) What is the gradient of a horizontal line?
(b) What name is given to a line that is perpendicular to a horizontal line?
(c) Why is the gradient of the line in (b) undefined?

odd 3 Give the gradients of lines *perpendicular* to lines with these gradients:

(a) p
(b) ^{-}p
(c) $\frac{1}{q}$
(d) $\frac{^{-}1}{q}$
(e) $\frac{p}{q}$
(f) $\frac{^{-}p}{q}$
(g) 0
(h) undefined

4 Here are the equations of 10 lines. Find the gradient for each line and then write down all the pairs of perpendicular lines. For example, if the gradients of f and j multiply to $^{-}1$, then one pair of perpendicular lines is $\{f, j\}$.

(a) $y = \frac{2x}{3} - 2$
(b) $y + 5x + 2 = 0$
(c) $2y + 3x = 8$
(d) $4y - 3x = 16$
(e) $4y = 3x - 12$
(f) $y = \frac{x}{5} + 3$
(g) $3y + 4x = 6$
(h) $y = \frac{^{-}3x}{2} + 1$
(i) $3y + 4x - 1 = 0$
(j) $y + 5x = 1$

5 Find the equation of the line through $(^{-}2, 3)$ perpendicular to the line $y = 2x + 4$.

6 Find the equation of the line through (6, 0) perpendicular to the line $y = \frac{3x}{4} + 2$.

7 Find the equation of the line through $(5, ^{-}1)$ perpendicular to:
(a) $x + 3y = 4$
(b) $2y = 6 - 3x$

8 Find the equation of the line through (1, 1) perpendicular to:
(a) $x = 4$
(b) $y = ^{-}5$

9 Two lines are perpendicular and intersect on the x-axis. One of the lines is $y = 2x - 6$. Find the equation of the other line.

10 Find the equation of the perpendicular bisector of the line joining (5, 1) and $(^{-}3, 7)$.

11 Find the equation of the line through A $(9, ^{-}2)$ perpendicular to the line joining A to the point $(^{-}1, 1)$.

12 Find the equation of the line through (0, 7) perpendicular to the line joining $(4, ^{-}1)$ to (6, 0).

13 Show that the triangle XYZ with X (4, 15), Y $(^{-}1, 4)$ and Z (7, 7) is right-angled, and find the equation of the hypotenuse.

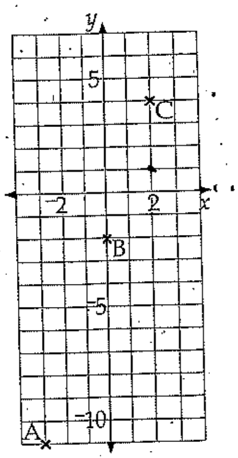
14 Determine the values of x that would make the points $(x, 0)$, (2, 1), and (6, 4) the vertices of a right-angled triangle.

Collinear points

Points are **collinear** if they all lie on the same line. This means that there should be no 'bend' as the points are joined, and, in particular, the gradient between any two pairs of points should be the same.

Example

Show that the points A $(^{-}3, ^{-}11)$, B $(0, ^{-}2)$ and C (2, 4) are collinear.



Answer

Do this question by showing that gradient AB = gradient BC.

$$\text{Gradient AB} = \frac{^-2 - {}^-11}{0 - {}^-3} = \frac{9}{3} = 3$$

$$\text{Gradient BC} = \frac{4 - {}^-2}{2 - 0} = \frac{6}{2} = 3$$

From A to B to C there is no change in the gradient, and so the three points A, B and C are collinear.

EXERCISE 16.12

1 Show that the points (2, 3), (4, 8) and (8, 18) are collinear.

2 Show that the points ($^-3$, 7), (0, 6) and (6, 4) are collinear.

3 A = (3, 4), B = (2, $^-1$) and C = (0, p). If A, B and C are collinear calculate the value of p.

4 A = ($^-1$, 1), B = (8, 3), C = (9, 7) and D = (0, 5).
  (a) Show that ABCD is a parallelogram.
  (b) What are the co-ordinates of the midpoint of AC?
  (c) Does the midpoint of AC lie on BD?

Drawing lines

Two pieces of information are usually needed to draw a line. These two can be given in a number of ways. For example:

1 one point on the line, and the gradient
2 two points on the line.

In both cases we start by having the equation of the line, and then obtain information from it before drawing the line.

1 Gradient–intercept method

If the equation is given in the form:

$$y = mx + c$$

then m is the gradient and c is the y-intercept, the place where the line cuts the y-axis.

To draw a line using this method we locate the y-intercept. Make sure that the equation is in the form $y = mx + c$ first. Then, from the y-intercept, draw a line that has a gradient of m. In general, this means 'go along one to the right, and then up or down by m'—up if m is positive: down if m is negative.

If the gradient is a fraction, it is sometimes easier to draw a line with this gradient by going along to the right by the denominator (bottom line) of the fraction, and then up or down by the numerator (top line).

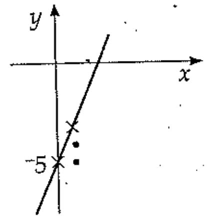
Example

Draw the line $y = 2x - 5$.

Answer

The equation is already in the form $y = mx + c$.

$m = 2 \qquad c = {}^-5$



The y-intercept is $^-5$, i.e. (0, $^-5$). Mark this point with a cross.

From the cross, draw a line with a gradient of 2 (i.e. out 1, up 2).

Mark this point with another cross. Join the two points up, extending the line in both directions.

Example

Draw the line $3x + 4y - 8 = 0$.

Answer

First rearrange to the form $y = mx + c$.

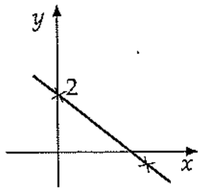
$$3x + 4y - 8 = 0$$
$$4y = {}^-3x + 8$$
$$y = \frac{^-3x}{4} + \frac{8}{4}$$
$$y = \frac{^-3x}{4} + 2$$

$$m = \frac{^-3}{4} \qquad c = 2$$

The y-intercept is 2, i.e. (0, 2). Mark this point with a cross.

From the cross, draw a line with a gradient of $\frac{^-3}{4}$ (i.e. out 4, down 3). Mark this point with another cross.

Join the two points up, extending a line in both directions.



Two special cases are **vertical lines** and **horizontal lines**.

- Vertical lines have equations of the form $x = c$. The line passes through the point $(c, 0)$.
- Horizontal lines have equations of the form $y = c$. The line passes through the point $(0, c)$.

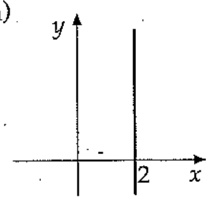
Example

Draw the lines:

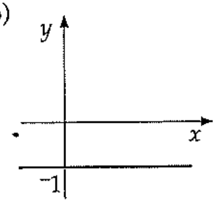
(a) $x = 2$ (b) $y = {^-1}$.

Answer

(a)



(b)



EXERCISE 16.13

Draw the following lines.

1 $y = 3x - 1$

2 $y = 3 - 2x$

3 $y = \frac{x}{3} + 2$

4 $y = 1 - x$

5 $y = x$

6 $y = \frac{2x}{3} - 4$

7 $y = \frac{^-3x}{4} + 3$

8 $y = \frac{x}{2}$

9 $y = \frac{x+1}{2}$

10 $y = \frac{x}{3}$

11 $y = 5x - 2$

12 $y = {^-x}$

13 $y = 4x - 3$

14 $y = 6$

15 $y = {^-2}$

16 $x = {^-4}$

17 $x = 3$

18 $y = \frac{3x-2}{4}$

19 $y = \frac{2-x}{2}$

20 $y = \frac{6-2x}{3}$

EXERCISE 16.14

Rearrange these equations to the form $y = mx + c$, and then draw each one.

1 $2y = 3x - 4$

2 $3y - 2x = 6$

3 $x + y + 4 = 0$

4 $x = y$

5 $3y - x = 6$

6 $4 - y = x$

7 $3x - 2y = 4$

8 $x + 6y = 12$

9 $5y + 2x = 10$

10 $3y = 3 - 6x$

11 $2y = 2 - x$

12 $2x - 3y = 6$

13 $4x + y = 5$

14 $2x + y + 3 = 0$

15 $2x + 3y + 9 = 0$

16 $x - y = 1$

17 $4x + 6 = 0$

18 $y - 5 = 0$

19 $\frac{x}{2} = 1$

20 $\frac{y-5}{3} = 1$

2 Intercept–intercept method

When the line equation is given in the form

$$ax + by + c = 0$$

it is usually easier to find the x and y intercepts and join them to form the line, than to rearrange and work out the gradient and y-intercept.

- The **x-intercept** is the point where the line cuts the x-axis. The co-ordinates of any point on the x-axis are $(p, 0)$. In other words, at the x-intercept, the y-value = 0.
- Similarly, the **y-intercept** is of the form $(0, q)$— i.e. at the y-intercept, the x-value is 0.

This suggests the following approach. Given a line equation in the form $ax + by + c = 0$:

1 To work out the x-intercept, substitute $y = 0$, and solve the equation for x.

2 To work out the y-intercept, substitute $x = 0$, and solve the equation for y.

Example

Find the x and y intercepts for the line $3x - 2y + 12 = 0$, and hence draw the graph.

Answer

For x-intercept substitute $y = 0$:

$$3x - 2 \times 0 + 12 = 0$$
$$3x = {}^{-}12$$
$$x = {}^{-}4$$

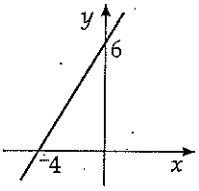
i.e. $({}^{-}4, 0)$ is the x-intercept.

For y-intercept substitute $x = 0$:

$$3 \times 0 - 2y + 12 = 0$$
$${}^{-}2y = {}^{-}12$$
$$y = 6$$

i.e. $(0, 6)$ is the y-intercept.

Plot these two intercepts, join them up, and extend the line in both directions.



EXERCISE 16.15

For each of the following equations, calculate the co-ordinates of the x and y intercepts, and hence draw the lines.

1 $2x + 3y = 6$
2 $4x - y = 4$
3 $2x - 5y = 10$
4 $3x - 2y = 6$
5 $4x = 2y + 12$
6 $3y = 6 - x$
7 $3x + 4y - 12 = 0$
8 $x - y - 6 = 0$
9 $4x + 6y - 5 = 0$
10 $2x - 5y + 4 = 0$

Writing the equation of a line given the graph

If you are given the graph of a line, there should always be enough information on the diagram to identify *two* points on the line.

Once the two points are known, it is best to use the **point/point equation**. An expression for this equation is:

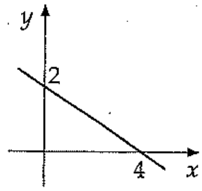
$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

How is this equation formed?

We are making two equivalent gradient statements. They must be equal to each other. Creating a single statement from these gradient expressions gives us an equation.

Example

Write down the equation of the line shown in the diagram.



Answer

Clearly, two points on the line are the x-intercept $(4, 0)$ and the y-intercept $(0, 2)$.

The equation is:

$$\frac{y - 0}{x - 4} = \frac{2 - 0}{0 - 4}$$

$$\frac{y}{x - 4} = \frac{2}{{}^{-}4}$$

$${}^{-}4y = 2(x - 4) \quad \text{(cross-multiplying)}$$

$${}^{-}4y = 2x - 8$$

$${}^{-}4y - 2x + 8 = 0$$

$$2x + 4y - 8 = 0$$

$$x + 2y - 4 = 0$$